5 Graphical Models

What You'll Learn

To use graphs to model real-world situations and data, and to compare, interpret, and analyse graphs

And Why

Recognizing trends and patterns in real-world graphs helps us better understand the situations being modelled, and allows us to make predictions about future behaviour.

Key Words

- trends
- rate of change
- first differences
- linear regression
- quadratic regression
- exponential regression

CHAPTER 5

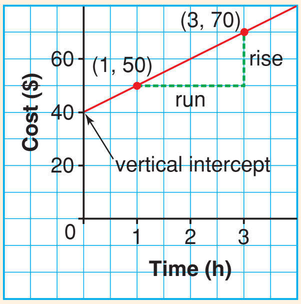
Activate Prior Knowledge

Linear, Quadratic, and Exponential Graphs

Prior Knowledge for 5.3, 5.4, and 5.5

The graph of a **linear relation** is a straight line.

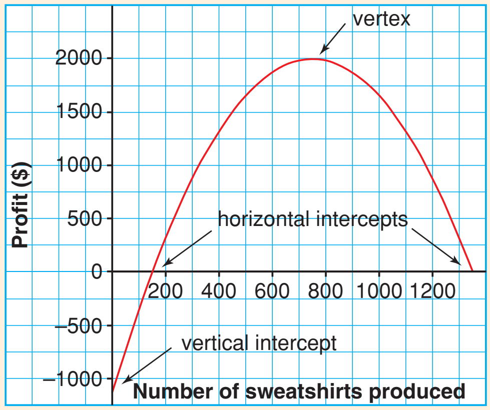
Cost of Magician



- Slope $= \frac{\text{rise}}{\text{run}}$

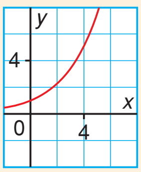
The graph of a **quadratic relation** is a **parabola**.

Sweatshirt Profits

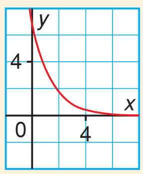


The graph of an **exponential relation** is an exponential curve.

Exponential Growth



Exponential Decay



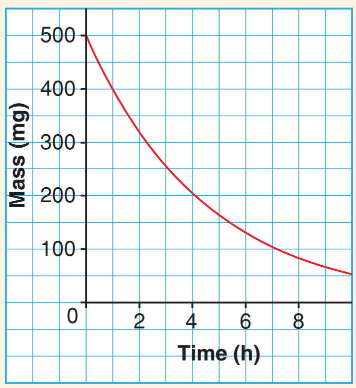
Example

This exponential graph shows the mass of pain medication left in a person's bloodstream after a tablet is swallowed.

a) How much medication was initially taken?

b) Determine the ratio of the mass of medication after 1 h to the mass of medication after 0 h. What does this value represent?

Pain Medication in the Body



Solution

a) The vertical intercept represents the mass of medication initially taken, 500 mg.

b) $$\text{Decay factor} = \frac{\text{Mass of medication after 1 h}}{\text{Mass of medication after 0 h}}$$
$$= \frac{400 \text{ mg}}{500 \text{ mg}}$$
$$= 0.8$$

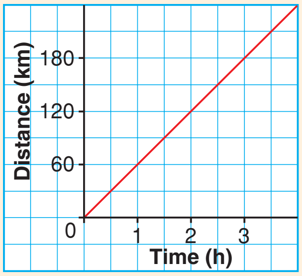
So, each hour, the mass of medication remaining in the body decreases by a factor of 0.8.

We can find the growth or decay factor by calculating the ratio of any two successive *y*-values. In this graph, the masses after 0 h and 1 h are easiest to read.

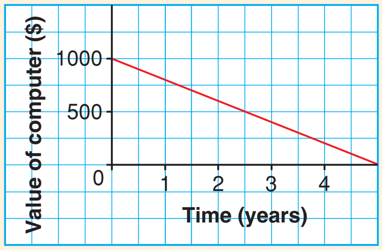
CHECK ✓

1. Determine the slope of each line. What does each slope represent?

a) **Distance Travelled by a Car**

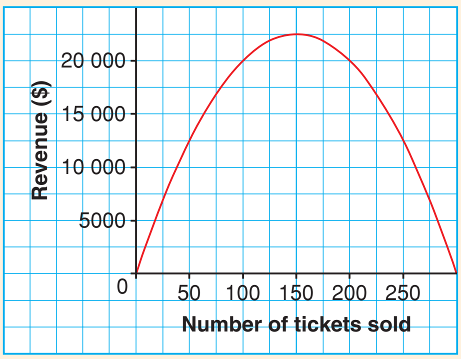


b) **Depreciation of the Value of a Computer**

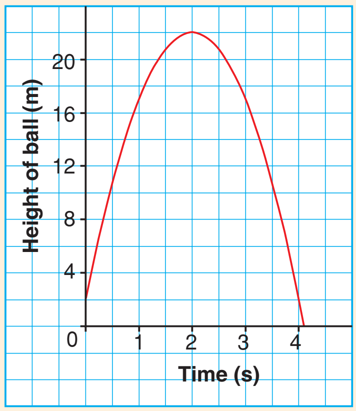


2. Determine the coordinates of the vertex of each parabola. What do these coordinates represent in each situation?

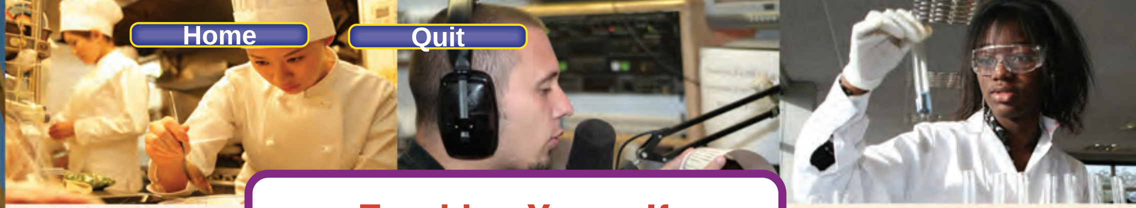
a) **Ticket Revenue**



b) **Height of Ball**



3. An exponential pattern of change results when an initial value is repeatedly multiplied by a constant factor. What role does each play in the graph?



Teaching Yourself

The ability to teach yourself is key to success in school, apprenticeships, and occupations. One of the best ways to do this is to learn to be an active reader.

Some strategies for active reading can be used in any subject.

- Highlight, underline, and make notes as you read. Identifying key information will make it easier to review.
- Use the glossary or a dictionary, or search on the Internet, to read about unfamiliar terms.
- Connect new concepts to ones you already know.
- Put the information into your own words, or create a graphic organizer such as a concept map, to show how pieces of information fit together.
- When possible, read information several times during a few days to allow time for reflection and to improve understanding.

Some additional strategies are needed for mathematics.

- Put yourself in the problem. For example, in the *Investigate* for Lesson 5.1, think about understanding the situation. Focus on the change in the global average temperature and carbon dioxide levels.
- Make a model, sketch, or diagram to help clarify your ideas.
- Do the math! If there is an example, you might rewrite the solution without looking at the book, then check.
- After you understand the example or question, try another question from the book, or make up a question yourself.

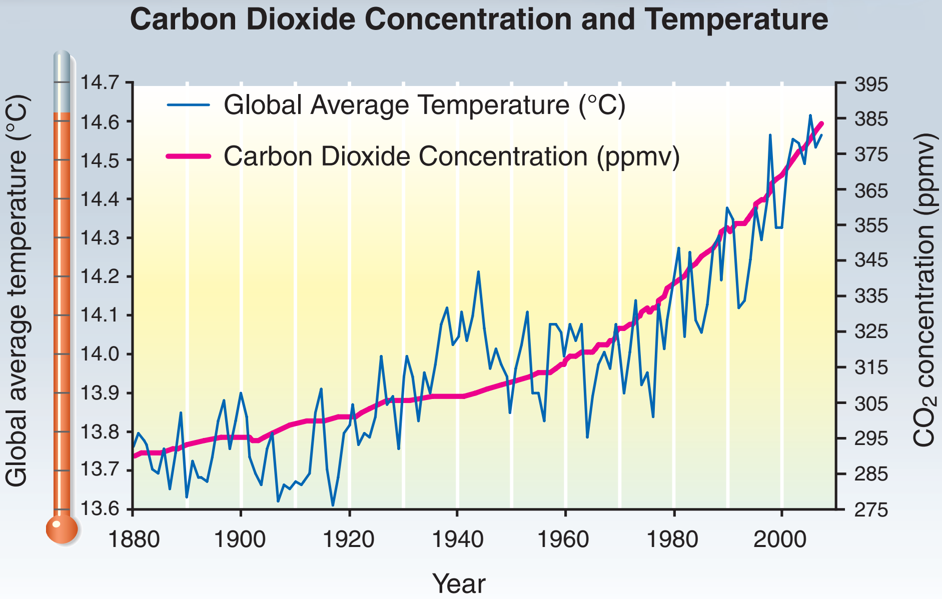
• Complete questions 1 and 2 as you complete the lessons.

1. Apply active reading as you work with graphing calculators and use regression to model data in the *Investigates* for Lessons 5.3 to 5.6 and the *Inquire* in Lesson 5.7.

2. Imagine being an apprentice, an employee, or a college student. How do you think you might use active reading in this role?

5.1 Trends in Graphs

Environmental scientists study issues such as global warming, air and water pollution, and resource management. Their work includes identifying trends in data and using these trends to predict future change.



Investigate

Analysing a Climate Change Graph

Work with a partner.

The graph above shows the change in global average temperatures and carbon dioxide (CO_2) levels in Earth's atmosphere from 1880 to 2007.

- What information can you read from the graph about global average temperatures? About CO_2 levels?
- What can you predict about global average temperatures and CO_2 levels over the next 20 years? How confident can you be about your predictions?

Reflect

- What factors did you consider in making your predictions?
- What factors may affect the reliability of your predictions?
- Does the graph show that the increase in CO_2 levels causes global warming? Justify your answer.

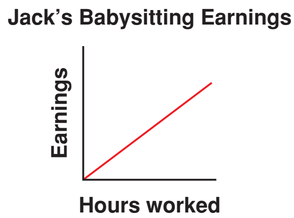
Connect the Ideas

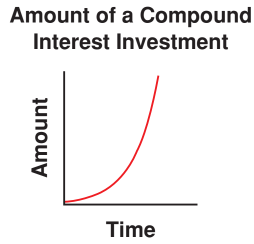
Using graphs to visualize relationships

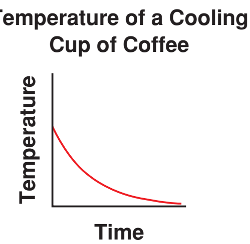
A graph is a visual representation of the relationship between two quantities. It shows how one quantity changes with respect to the other.

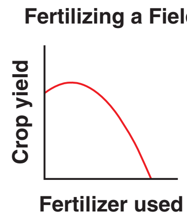
Example 1 Describing Relationships in Graphs

Describe the relationship shown in each graph.

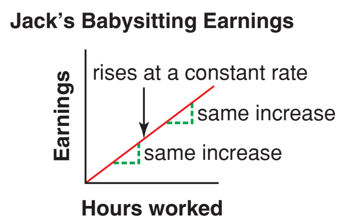
a) 

b) 

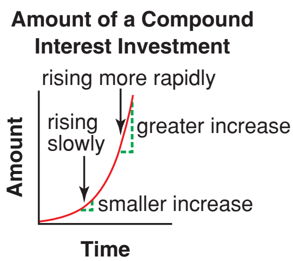
c) 

d) 

Solution

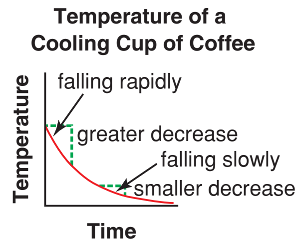
a) 

Pairs of points with equal horizontal distances have equal vertical distances. As the number of hours Jack works increases, his earnings increase by a constant amount.

b) 

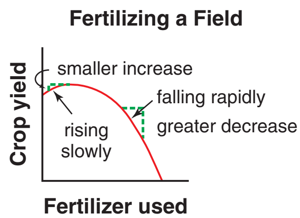
The vertical distances between pairs of points with equal horizontal distances are increasing. The amount of the compound interest investment increases over time, slowly at first and then more rapidly.

c)



The vertical distances between pairs of points with equal horizontal distances are decreasing. The coffee temperature decreases over time, rapidly at first, then more slowly, and finally levelling off at room temperature.

d)



The vertical distances between pairs of points with equal horizontal distances are decreasing, then increasing. As fertilizer use increases, the crop yield increases, reaches a maximum, and then decreases.

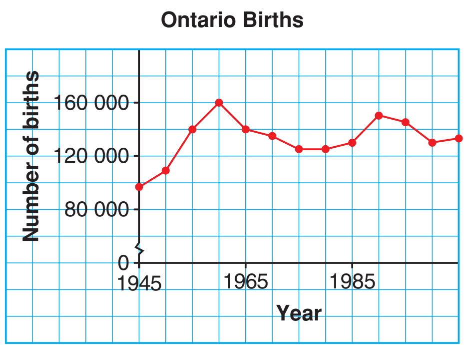
Trends in graphs

Trends, or patterns of change, in a graph are often used to justify decisions and make predictions.

Example 2

Describing the Trends in a Graph

This graph shows the number of births in Ontario from 1945 to 2005. Describe the trends in the graph.



Solution

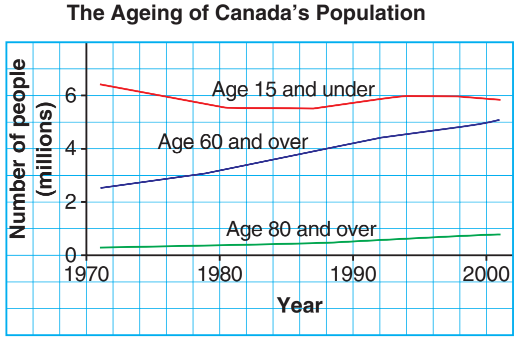
Trends occur in 3 broad groups: increasing, decreasing, and constant (no change). Divide the graph into intervals of time when the number of births is increasing, constant, or decreasing.

- From 1945 to 1960, the number of births is increasing rapidly. There is a maximum number of births in 1960.
- From 1960 to 1975, the number of births is decreasing, rapidly at first, then slowly, then more rapidly again.
- From 1975 to 1980, the number of births is constant.
- From 1980 to 1990, the number of births is increasing, slowly at first, then rapidly. There is another maximum of births in 1990.
- From 1990 to 2000, the number of births is decreasing rapidly.
- From 2000 to 2005, the number of births is increasing slowly.

Example 3 Using Trends to Make Predictions and Justify Decisions

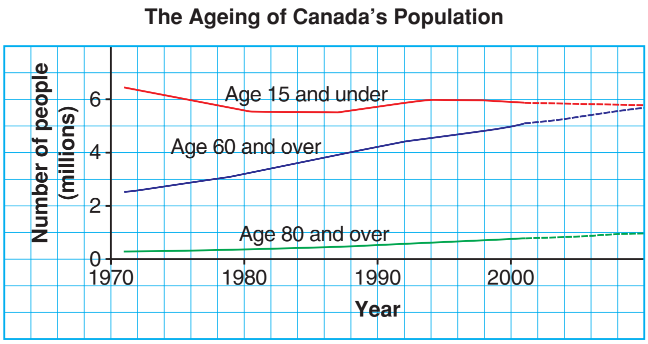
a) Use the graph to predict the number of Canadians in each age group in 2010.

b) What decisions might the Canadian government make in response to the trends in the graph?



Solution

a) Continue the trends to 2010.



Record the projected number of Canadians in each age group.

- Under age 15: About 5.8 million
- Age 60 and over: About 5.7 million
- Age 80 and over: About 1.0 million

b) The trends suggest that a declining number of younger Canadians will have to support an increasingly larger number of elderly Canadians. Some decisions the Canadian government may make in response are:

- Increasing immigration levels to prevent future labour shortages
- Increasing the retirement age
- Strengthening health care and social security to better address the needs of older Canadians

Example 3 illustrates that often the best prediction we can make is to continue the trend. However, only a short-term prediction is reliable because we cannot be certain that the trend will continue, or, there may be several reasonable ways to continue the trend.

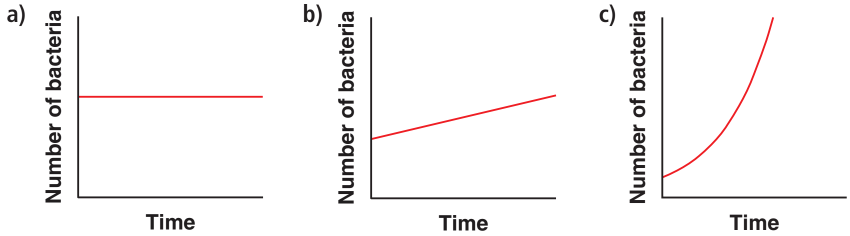
Practice

In questions 1 to 4, choose the graph that best represents the given description. Justify your choice.

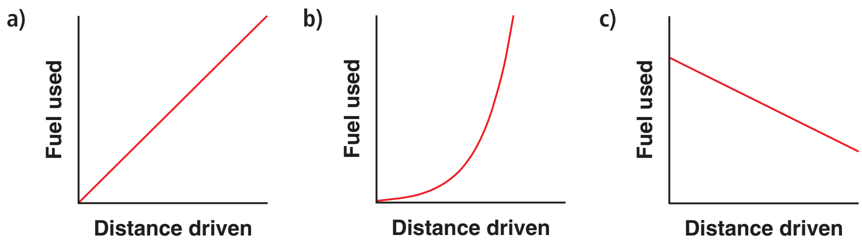
A

For help with question 1 to 4, see Example 1.

1. The number of bacteria in a laboratory colony increases over time, slowly at first and then more rapidly.

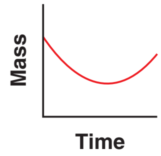


2. The fuel used increases steadily as the distance driven increases.



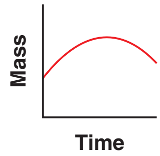
3. As the price increases, the revenue earned increases, reaches a maximum, then decreases.

4. The radioactive substance decayed over time, rapidly at first, then more slowly.

a)


b)


c)


d)


5. Match each graph with the statement that best describes it. Which words gave clues about the shape of the graph?

i)


ii)


iii)


iv)


a) Sales have fallen dramatically over the last year.

b) Sales have fallen steadily over the last year.

c) Sales have remained constant over the last year.

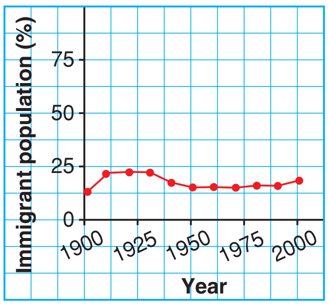
d) Sales have fluctuated over the last year.

B

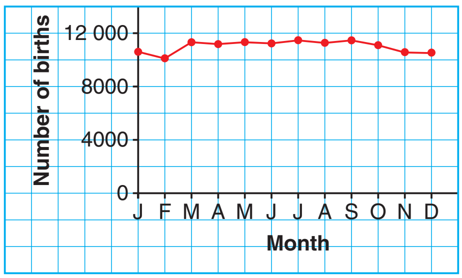
■ For help with question 6, see Example 2.

6. Describe the trends in each graph.

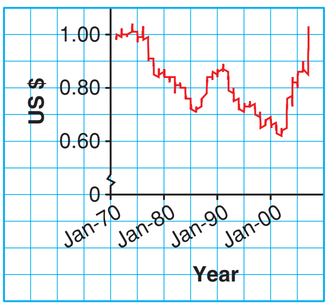
a) **Percent of Immigrants in the Population of Canada**



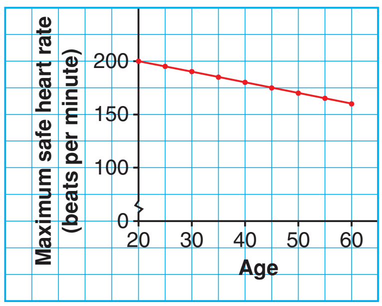
b) **Number of Births in Each Month in 2004**



c) **Value of $1Can in US Dollars**

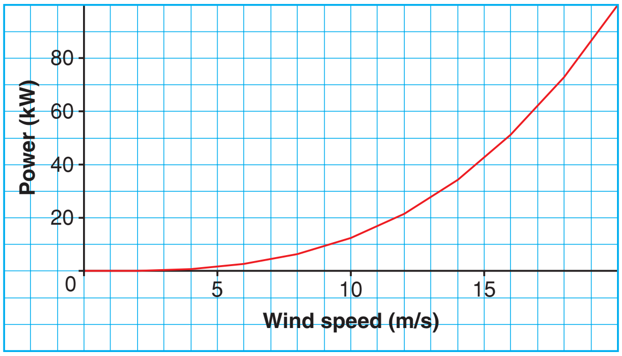


d) **Maximum Safe Heart Rate During Exercise**



7. Wind power is becoming a practical source of renewable energy. This graph shows how the power generated by a wind turbine with radius 5 m changes with the wind speed.

Power Generated by a Wind Turbine



a) Describe the general trends in the graph.

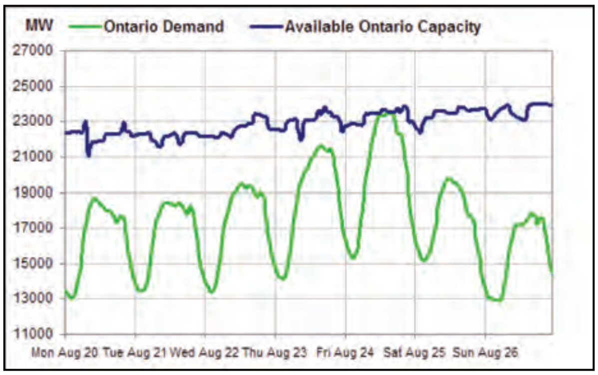
b) What is the vertical intercept? What does it represent in this situation?

c) How much power is generated when the wind speed is 20 m/s?

d) Does the power generated double when the wind speed doubles? Explain.

8. Each Tuesday, the Independent Electricity System Operator publishes a summary of the power generated and used in Ontario in the previous week.

Week ending August 26, 2007	
Ontario Peak Demand	**(MW)**
4 p.m. to 5 p.m. August 24, 2007	23,497
Wholesale Prices	**(¢ per kWh)**
Average Weekday Prices (8 a.m. to 8 p.m.)	6.41
Average Prices (other times)	4.72
(Prices weighted by Ontario Demand)	



a) Describe the trend of Ontario electric power capacity for the week of August 20–26, 2007.

b) Explain the daily pattern of the Ontario electric demand.

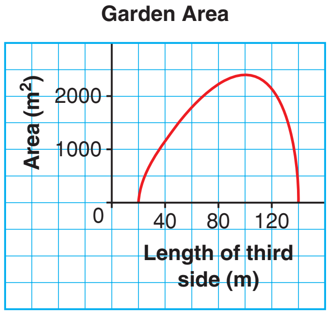
c) How does peak daily demand change over the week? Why might this happen?

d) When does demand increase to or above capacity? How do you know?

e) What happens to electric service when demand surpasses capacity?

■ For help with question 9, see Example 3.

9. You design a triangular garden for a park. One side of the garden has length 60 m and another side has length 80 m. This graph shows how the area of the garden is related to the length of the third side.



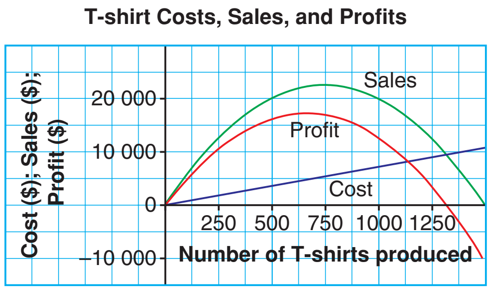
a) Describe the relationship between the length of the third side and the area of the garden.

b) How could you use the graph to decide what the length of the third side should be in each situation? Justify your answers.

i) You want the area of the garden to be 1500 m^2.

ii) You want the garden to have the maximum possible area.

10. A small company makes and sells T-shirts. This graph shows how the cost, sales, and profit vary as the number of T-shirts produced increases.



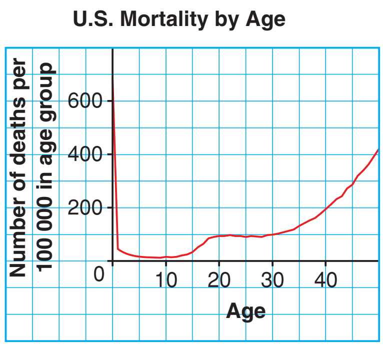
a) Describe each relationship. Explain your reasoning.

i) The relationship between the cost and the number of T-shirts produced.

ii) The relationship between the profit and the number of T-shirts produced.

b) Describe at least 2 decisions that the sales manager of the company might make based on the trends in the graph. Justify your answers.

11. **Assessment Focus**



a) Describe the trends in this graph.

b) What is the minimum point on the graph? What do its coordinates represent?

c) Describe the trend in mortality from age 15 to age 20. Why do you think this occurs? What decisions may be made in response to this trend? Explain.

d) Predict the mortality rate at age 70. Justify your prediction.

e) The actual mortality rate at age 70 is 1754 deaths per 100 000 seventy-year-olds. Compare your predicted result to this result. What might account for differences between the actual and predicted values?

12. **Literacy in Math** Use a matrix or another graphic organizer to summarize the important features of a graph. Some possible headings for a matrix are given below.

	Intercepts	Maximum or minimum points	Trends
Definition			
How to recognize			
Example			

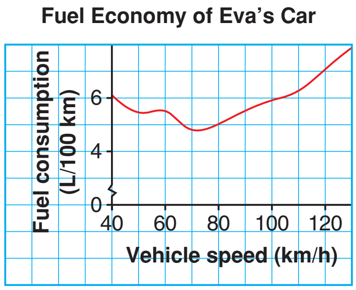
C

13. This graph shows the fuel economy of Eva's car at various speeds.

a) Describe the relationship between speed and fuel economy.

b) Use the graph and the current cost of fuel. How much money would Eva save on a 1200 km trip by driving at the speed that produces the greatest fuel economy instead of the speed limit of 100 km/h? Justify your answer.

c) How much longer will the trip in part b take at the speed with the greatest fuel economy?



14. **a)** Describe the general trend in this graph.

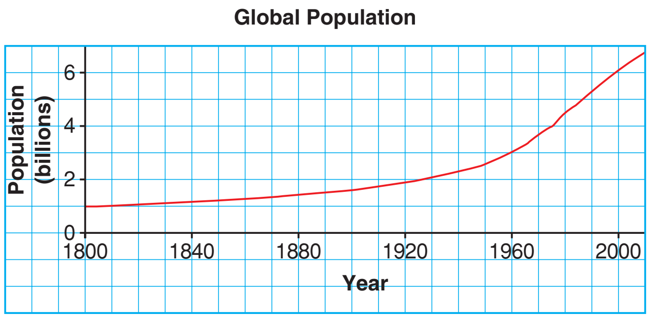
b) Predict when the world population will reach 9 billion under each scenario.

i) The current rate of growth continues

ii) The growth rate decreases

iii) The growth rate increases

Explain how you made your predictions.



In Your Own Words

Suppose you collect data about the mass of a puppy over 15 weeks and plan to present the data in a graph. What trends would you expect to see in the graph? Explain.

5.2 Rate of Change

A child's height and growth rate are important indicators of the child's overall health.

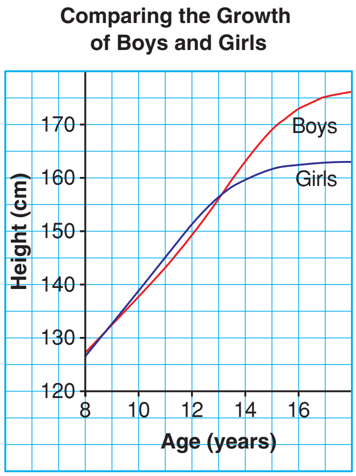
Investigate

Analysing Patterns of Growth in Girls and Boys

Work with a partner.

This graph presents data from the World Health Organization on average heights of boys and girls.

- What information can you read from the graph? Record your ideas.



The World Health Organization is an agency of the United Nations.

Reflect

- How does a graph communicate information about change?
- Compare the information you read from the graph with another pair of students. What additional information did they find?

Connect the Ideas

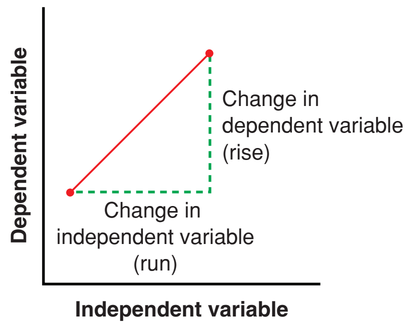
Rate of change

We can determine an average rate of change using a table or graph.

Table

Independent variable	Dependent variable
x_1	y_1
x_2	y_2

Graph



The average rate of change between two points is the slope of the line segment joining the points.

Average rate of change

$$\text{Average rate of change} = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{\text{Rise}}{\text{Run}}$$

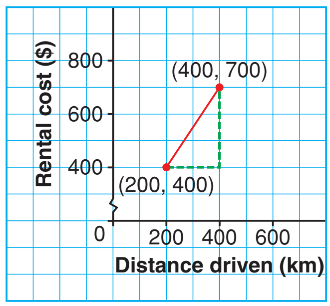
Example 1 Calculating and Interpreting Rates of Change

Calculate the average rate of change between each pair of points. Explain what the rate of change represents.

a)

Time (min)	Height of airplane (m)
0	2000
4	1400

b) Vehicle Rental Cost



Solution

a) Average rate of change

$$= \frac{\text{Change in height}}{\text{Change in time}}$$

$$= \frac{1400 \text{ m} - 2000 \text{ m}}{4 \text{ min} - 0 \text{ min}}$$

$$= \frac{-600 \text{ m}}{4 \text{ min}} \text{ or } -150 \text{ m/min}$$

The height of the airplane is decreasing by an average of 150 m each minute.

b) Average rate of change

$$= \frac{\text{Change in cost}}{\text{Change in distance driven}}$$

$$= \frac{\$700 - \$400}{400 \text{ km} - 200 \text{ km}}$$

$$= \frac{\$300}{200 \text{ km}} \text{ or } \$1.50/\text{km}$$

The cost increases by an average of $1.50 for each kilometre driven.

Example 2 Comparing Rates of Change

The *reaction distance* is the distance the car travels from the time the driver decides to stop the car until the driver applies the brakes. The *braking distance* is the distance the car travels from the time the brakes are applied until the car stops.

The distance required to stop a car depends on the speed at which the car is travelling.

These tables show the reaction distance and braking distance needed to stop a car on dry pavement for given speeds.

Speed (km/h)	0	10	20	30	40	50
Reaction distance (m)	0	2	4	6	8	10

Speed (km/h)	0	10	20	30	40	50
Braking distance (m)	0.0	0.5	2.0	4.5	8.0	12.5

The sum of the reaction distance and braking distance is the *stopping distance*.

a) Calculate the average rate of change between consecutive points in each table. Describe the rates of change revealed in each table.

b) Graph the data in the tables. Describe how the graph reflects the rates of change across the data.

Solution

a)

Speed (km/h)	Reaction distance (m)	$\frac{\text{Change in distance}}{\text{Change in speed}}$
0	0	
		$\frac{2-0}{10-0}=\frac{2}{10}=0.2$
10	2	
		$\frac{4-2}{20-10}=\frac{2}{10}=0.2$
20	4	
		$\frac{6-4}{30-20}=\frac{2}{10}=0.2$
30	6	
		$\frac{8-6}{40-30}=\frac{2}{10}=0.2$
40	8	
		$\frac{10-8}{50-40}=\frac{2}{10}=0.2$
50	10	

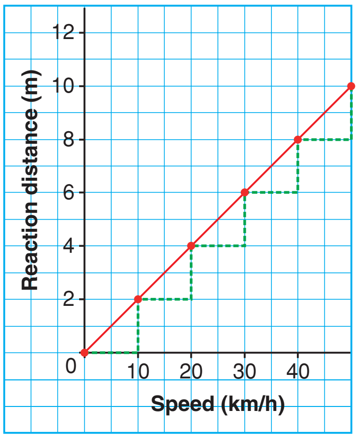
The rates of change are constant: 0.2. So, the reaction distance increases by 0.2 m for every 1-km/h increase in speed.

Home Quit

Speed (km/h)	Stopping distance (m)	$\frac{\text{Change in distance}}{\text{Change in speed}}$
0	0.0	
		$\frac{0.5-0.0}{10-0}=\frac{0.5}{10}=0.05$
10	0.5	
		$\frac{2.0-0.5}{20-10}=\frac{1.5}{10}=0.15$
20	2.0	
		$\frac{4.5-2.0}{30-20}=\frac{2.5}{10}=0.25$
30	4.5	
		$\frac{8.0-4.5}{40-30}=\frac{3.5}{10}=0.35$
40	8.0	
		$\frac{12.5-8.0}{50-40}=\frac{4.5}{10}=0.45$
50	12.5	

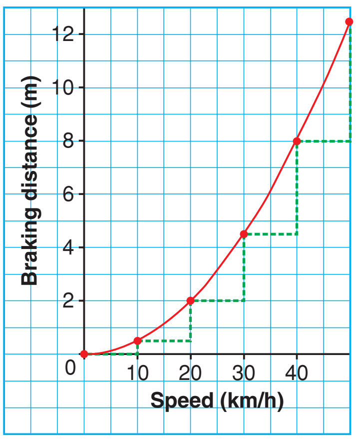
The rates of change are increasing.

b) Reaction Distance and Speed



Since the rate of change is the same in each 10 km/h interval, the points on the graph lie on a straight line.

Braking Distance and Speed



Since the average rate of change is different for each 10 km/h interval, the points on the graph do not lie on a straight line.

Since a linear relation has a constant rate of change, we refer to the *rate of change* of a linear relation and the *average rate of change* of a non-linear relation.

Example 3 Identifying Rates of Change in a Table and Graph

This table shows the change in height of a tomato plant from germination until the tomatoes ripen.

Time (weeks)	0	2	4	6	8	10	12	14	16	18
Height (cm)	0	5	10	20	40	58	75	86	90	90

a) Determine when the rate of change in the height is:

i) Zero **ii)** Constant **iii)** Changing

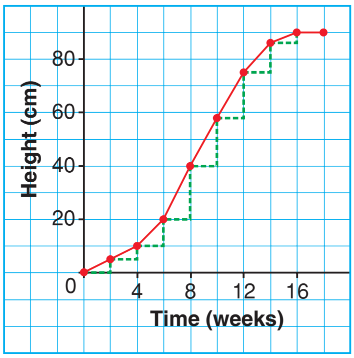
b) When is the rate of change in height the greatest?

c) Describe the growth of the plant.

Solution

a) Create a table of first differences. Graph the data from the table.

Growth of Tomato Plant



Time (weeks)	Height (cm)	First differences
0	0	
		5 − 0 = 5
2	5	
		10 − 5 = 5
4	10	
		20 − 10 = 10
6	20	
		40 − 20 = 20
8	40	
		58 − 40 = 18
10	58	
		75 − 58 = 17
12	75	
		86 − 75 = 11
14	86	
		90 − 86 = 4
16	90	
		90 − 90 = 0
18	90	

On the graph, the vertical distances between consecutive points correspond to the first differences.

i) In the table, look for a first difference of 0. The rate of change is zero from week 16 to 18.

ii) On the graph, look for points lying along a line. The rate of change is constant from week 0 to 4.

The vertical distances between consecutive points on a line are constant.

iii) The rate of change is changing when the vertical distances between consecutive points change. The rate of change is changing from week 4 to 16.

b) The rate of change is greatest from week 6 to 8.
This can be seen in the graph, where the steepest line segment is between these two weeks.

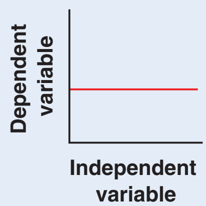
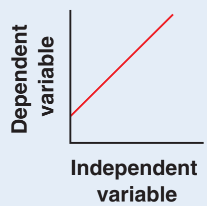
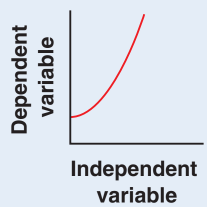
c) The plant grows at a constant rate for the first 4 weeks. The rate of growth increases over the next 4 weeks, and then decreases as the plant reaches maturity. After 16 weeks, the plant has reached its full height, and the rate of growth is 0.

Identifying when a rate of change is zero, constant, or changing

Given a table or graph, we can identify when the rate of change is zero, constant, or changing without actually calculating the rates of change.

- In a table, we look at the first differences.
- In a graph, we decide whether the points lie on a line.

Identifying Rates of Change in a Table or Graph

Rate of change	Table	Example of graph
Zero	The first differences are 0.	
Constant	The first differences are equal.	
Changing	The first differences are changing.	

Practice

A

For help with questions 1 to 5, see Example 1.

1. For each table, name the variables.

a)

Hours worked	Earnings ($)
4	32
20	160

b)

Pages printed	Cost ($)
1000	56
5000	145

c)

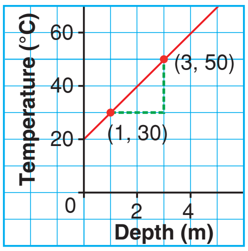
Distance driven (km)	Fuel used (L)
45	3
60	12

2. State the units of the rate of change for each table in question 1. What does the rate of change represent?

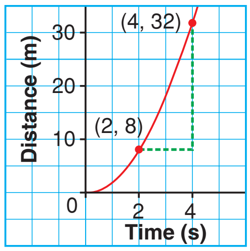
3. Refer to the tables in question 1. Determine the average rate of change between each pair of points in the table.

4. For each graph, name the variables.

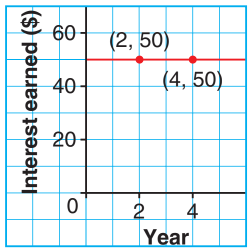
a) Graph A



b) Graph B



c) Graph C



5. State the units of the rate of change in each graph in question 4. What does each rate of change represent?

6. Refer to the graphs in question 4. Determine the average rate of change between the indicated points on the graph.

B

7. To save energy, an office building is only heated during business hours.

a) When is the temperature:

i) Constant?

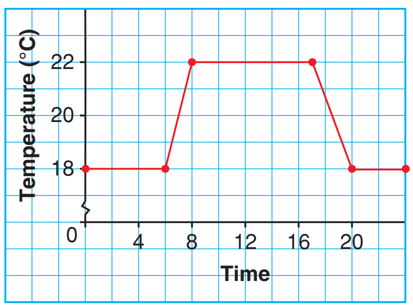
ii) Decreasing?

iii) Increasing?

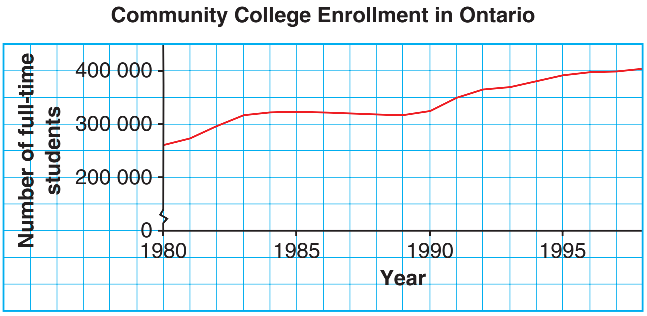
b) Calculate the rate of change during each time period from part a.

c) Describe the connection between your answers in parts a and b.

Office Building Temperature



8. **a)** Describe the trends in this graph.



b) From 1980 to 1998, the number of students attending a community college changed from 260 761 students to 403 516 students. What was the average rate of change in attendance during this time?

c) The graph is approximately horizontal from 1983 to 1989. What does this tell you about the rate of change in attendance during this time?

■ For help with question 9, see Example 2.

9. Bipin is a financial advisor. He uses these tables to help his clients understand the difference between simple interest and compound interest.

Simple Interest				
Year	0	5	10	15
Amount ($)	500	700	900	1100

Compound Interest				
Year	0	5	10	15
Amount ($)	500	735	1079	1586

a) Calculate the average annual rate of change for consecutive pairs of data in each table.

b) Describe the rates of change in each table. What do these values indicate about each type of interest?

c) Graph the data in the tables. How is the rate of change reflected in the graph?

10. **Assessment Focus**

A tanker runs aground, creating a circular oil spill.

a) For each graph, calculate the average rate of change:

  i) From 0 min to 5 min

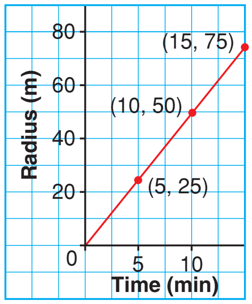
  ii) From 10 min to 15 min

  What do the rates of change represent in this situation?

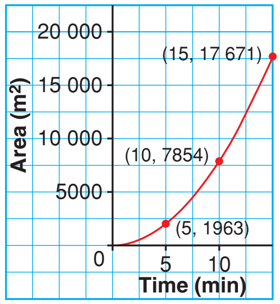
b) Describe the change in the radius of the spill.

c) Describe the change in the area of the spill.

Radius of Oil Spill



Area of Oil Spill



For help with questions 11 and 12, see Example 3.

11. Use this table to analyse changing power needs in Canada.

Total Amount of Electric Power Generated, Canada

Year	1950	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Amount (MWh)	64	79	108	149	209	285	360	433	490	530	564	578

a) Create a table of first differences. What do they represent in this situation?

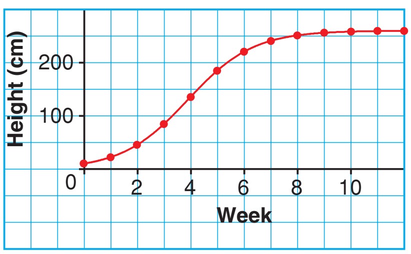
b) When did the electric energy generated increase most rapidly?

c) The first differences are approximately equal from 1975 to 1980 and from 1980 to 1985. What does this mean about the electric energy generated in those years?

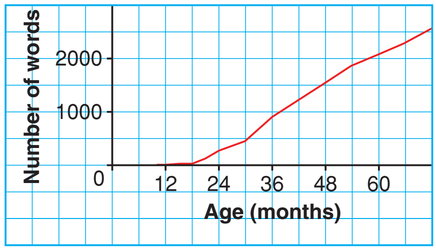
d) Describe how the electric energy generated has changed from 1950 to 2005.

12. The height of a sunflower was recorded every week over a growing season. Describe the growth of the sunflower.

Height of a Sunflower



13. Children learn new words at varying speeds. This graph shows a typical learning curve for children from ages 10 months to 72 months.



a) When are new words learned most quickly?

b) What is the average rate of learning at that time?

c) What is the average rate of learning from 60 to 72 months of age?

d) Suppose a child started elementary school at 72 months. Predict and sketch a continuation of the graph for another 72 months.

e) Explain the shape you chose for part d.

14. **Literacy in Math** You have described change and rates of change in Lessons 5.1 and 5.2. Use a graphic organizer of your choice to summarize some of the key ideas and vocabulary you have used in these lessons.

C

15. Refer to the graph for the growth of a sunflower in question 12.

a) Sketch a graph for a sunflower that takes the same time to mature but has half the final height. Explain your reasoning.

b) Sketch a graph for a sunflower that takes longer to mature but has the same final height. Explain your reasoning.

16. The temperature in a sunroom of a house changes throughout the day. Use the following information to draw a graph of the temperature over time.

- The early morning temperature is 12°C.
- After the sun reaches the sunroom at 8:00 A.M., the temperature rises 4°C/h until 12:00 P.M.
- The temperature is approximately constant from 12:00 P.M. in the afternoon until 3:00 P.M.
- After that it drops about 2°C/h until it reaches 12°C.

In Your Own Words

Draw or find a graph that shows how a quantity changes over time. Use words and numbers to describe the change and rates of change shown in the graph. What do these tell you about how the quantity is changing?

Home Quit

5.3 Linear Models

The chirping rate of a cricket increases as the temperature rises and decreases as the temperature falls. Biologists call crickets "nature's thermometers" because the chirp rates can be used to predict the temperature.

Investigate — Chirping Rates of a Cricket

Materials
- TI-83 or TI-84 graphing calculator

Work with a partner.

A biologist collects data about the temperature and chirp rate of a cricket.

Temperature (°F)	Chirp rate (chirps/min)
61	87
66	102
67	109
73	136
74	154
76	150
77	154
78	160

- What trends do you see in the data? How well does a line model the trends in the data? Justify your answers.
- Is it reasonable to use a linear model to predict the chirp rate when the temperature is:
  a) 70°F?
  b) 0°F?
  c) 120°F?
  Explain.

Reflect

What assumptions do we make when we create a line of best fit and use it to make predictions? Why might these assumptions not be valid?

Connect the Ideas

Mathematical models

Tables, graphs, and equations are examples of **mathematical models**. Mathematical models allow us to represent the relationship between real-world quantities, analyse current behaviour, and predict future behaviour.

Mathematical models can be numerical (tables), graphical (graphs), or algebraic (equations).

Linear models

Linear models

A linear model represents quantities that increase or decrease by a constant amount over equal intervals.

- In a table of values, the first differences are equal.
- The graph is a straight line.
- The equation of the line can be written in the form $y = mx + b$, where m is the slope and b is the vertical intercept.
- The rate of change is constant.

Example 1 Identifying Linear Models

Which models represent linear relations? Justify your answers.

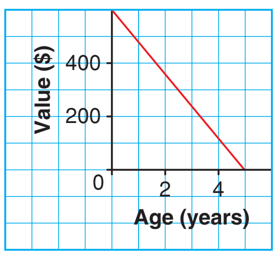
a)

Time (s)	0	1	2	3
Height (m)	60	55	40	15

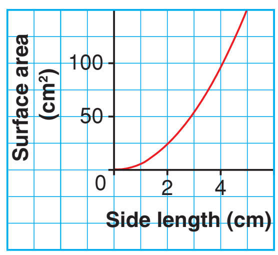
b)

Hours	0	5	10	15
Earnings ($)	0	40	80	120

c) Depreciation in Value of Printer



d) Surface Area of a Cube



e) $y = 2x + 5$

f) $y = x^2 + 5$

Solution

a)

Time (s)	Height (m)	First differences
0	60	
		$55 - 60 = -5$
1	55	
		$40 - 55 = -15$
2	40	
		$15 - 40 = -25$
3	15	

The first differences are not equal.
The relationship is non-linear.

b)

Hours	Earnings ($)	First differences
0	0	
		$40 - 0 = 40$
5	40	
		$80 - 40 = 40$
10	80	
		$120 - 80 = 40$
15	120	

The first differences are equal.
The relationship is linear.

c) The graph is a line.
The relationship is linear.

d) The graph is a curve.
The relationship is non-linear.

e) $y = 2x + 5$ is of the form $y = mx + b$. The relationship is linear.

f) $y = x^2 + 5$ cannot be expressed in the form $y = mx + b$. The relationship is non-linear.

Analysing the graph of a linear relation

In real-world graphs of linear relations:

- The vertical intercept represents the initial value of the dependent variable.
- The slope represents the rate of change in the dependent variable with respect to the independent variable.

We can compare the graphs of pairs of relations to investigate how the initial value and rate of change are reflected in the graph.

Example 2 Comparing Pairs of Linear Relations

Materials

- TI-83 or TI-84 graphing calculator

A cup of coffee is reheated in a microwave.
The temperature, C degrees Celsius, of the coffee after t seconds can be modelled by linear equations.

500-W power setting: $C = 0.5t + 20$

1000-W power setting: $C = t + 20$

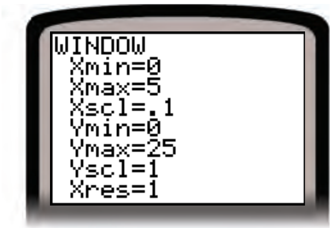
a) Explain what the numbers in the equations represent.

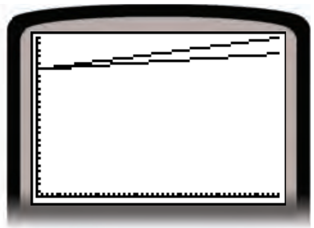
b) Graph both equations on the same screen of a graphing calculator. How are the numbers in the equation reflected in the graph?

Solution

a)
- The coefficients of t, 0.5 and 1, represent the rate of change of the temperature with respect to time. At the 500-W setting, the temperature increases at a rate of 0.5°C/s. At the 1000-W setting, the temperature increases at a rate of 1°C/s.
- The constant 20 represents the temperature of the coffee at $t = 0$ s. This is the temperature of the coffee when it was placed in the microwave.

b) Use a graphing calculator to generate the graphs of $y = 0.5x + 20$ and $y = x + 20$ on the same screen.





Both graphs have the same vertical intercept, but the line representing the 1000-W setting is steeper than the line representing the 500-W setting. This is because both cups of coffee have an initial temperature of 20°C, and the rate of change in temperature is greater at the 1000-W setting.

Fitting a linear model to data

We can use linear regression to model data that appear to be linearly related. The regression line can then be used to analyse the data and make predictions. The closer the regression line is to the data points, the more reliable the predictions are likely to be.

Example 3

Fitting a Linear Model to Data

Materials
- TI-83 or TI-84 graphing calculator

This table shows the median age of Canada's population from 1975 to 2000.

Year	1975	1980	1985	1990	1995	2000
Median age (years)	27.4	29.1	31.0	32.9	34.8	36.8

a) Create a scatter plot of the data and describe any trends.

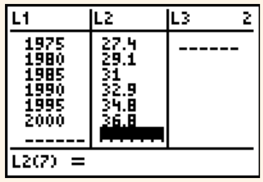
b) Determine the equation of the regression line. Graph the regression line on the scatter plot.

c) Predict the median age of Canada's population in 2020.

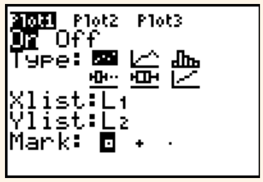
Solution

a) Create a scatter plot of the data.

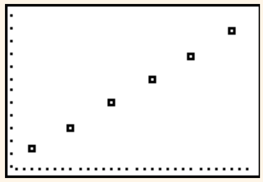
Enter the data in lists **L1** and **L2**.



Press [2nd] [Y=] 1 and change the settings as shown.



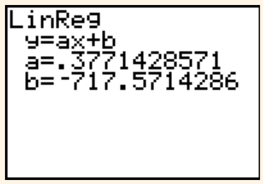
Press: [ZOOM] 9



The data appear to lie on a straight line. This suggests that the median age has been increasing at a constant rate.

b) Use the **LinReg(ax+b)** command.

Press [STAT] [▶] 4 [2nd] 1 [,] [2nd] 2 [,] [VARS] [▶] 1 1 [ENTER] to perform linear regression on the data and store the regression equation as **Y1**.



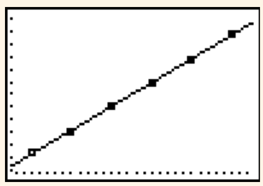
Since the points lie close to the line, the regression line is a good fit for the data.

The regression equation is given in the form $y = ax + b$, where a is the slope of the line and b is the y-intercept.

So, the regression equation is: $y \doteq 0.3771x - 717.5714$

Graph the data and the regression curve.

Press: [GRAPH]

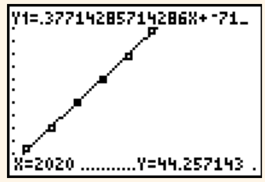


c) Here are two methods for predicting the median age in 2010.

Method 1: Using the regression line

Use the **TRACE** feature to estimate the median age of Canada's population in 2020.

Press WINDOW and set **Xmax** = 2020.
Press: TRACE
Press ▼ to place the cursor on the graph of the regression equation.
Press: 2020 ENTER



At $x = 2020$, $y \doteq 44.2$

Method 2: Using the regression equation

You could also use the command **Y1(2020)** on your graphing calculator. Press: VARS ▶ 1 1 (2020) ENTER

Substitute $x = 2020$ into the regression equation.

$y \doteq 0.3771x - 717.5714$
$\doteq 0.3771(2020) - 717.5714$
$\doteq 44.2$

If the current trends continue, the median age of Canada's population in 2020 will be about 44 years.

Practice

A

For help with questions 1 to 3, see Example 1.

1. Which tables of values model a linear relation? How do you know?

a)

r	0	5	2	3	4	5
C	0.0	31.4	62.8	94.2	125.7	157.1

b)

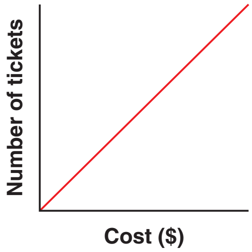
t	0	1	2	3	4	5
h	282.5	272.7	243.3	194.3	125.7	37.5

2. Which equations model a linear relation? How do you know?

a) $y = -2x$ **b)** $y = x^2 + 1$ **c)** $y = 5 - 2x$

3. Which graphs model a linear relation? How do you know?

a)



b)

c)

■ For help with questions 4 and 5, see Example 2.

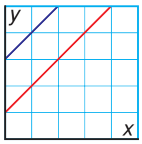
4. Match each graph with the statement that best describes it.

a) Same initial value, different rates of change

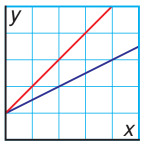
b) Different initial values, same rate of change

c) Different initial values, different rates of change

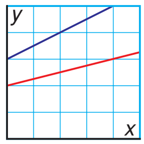
i)



ii)



iii)

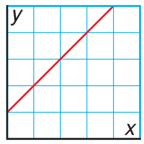


5. For each part, sketch a line that is different from this line.

a) Different initial value

b) Greater rate of change

c) Same initial value and a zero rate of change



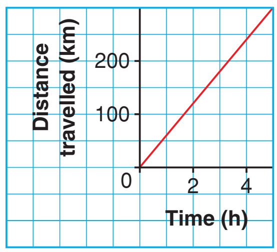
B

6. This graph shows how the distance a car travels changes over time.

a) Calculate the rate of change. Does it matter which points you use? Explain.

b) What does the rate of change represent? How do you know?

Distance Travelled by a Car



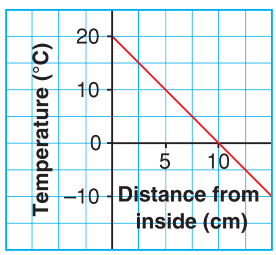
7. An energy auditor uses a temperature probe to check the insulation in a wall of a house.

a) Describe the rates of change revealed in the graph.

b) Calculate the rate of change in temperature with respect to distance into the wall.

c) Write an equation that describes the temperature-distance relation.

Insulation Temperature at Various Distances from Interior Wall



8. Two springs are stretched to determine which one is stiffer. The force, F newtons, needed to stretch the springs a distance of x cm is given by these formulas.

Spring 1: $F = 0.1x$ **Spring 2:** $F = 0.5x$

a) Describe what the graph would look like if you graphed both equations on the same grid. What would be the same about both graphs? What would be different? How do you know?

b) Use graphing technology to verify your answer to part a.

c) Determine the rate of change in force with respect to distance for each spring. Explain your method.

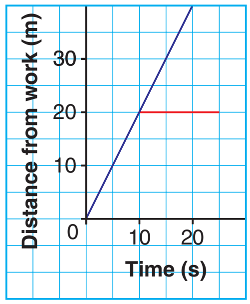
d) The rate of change in force for a spring is called the *spring constant*. If a spring is easy to stretch, does it have a high or low spring constant? Explain.

9. Two friends go for a walk during their lunch break. This graph shows how their distance from work changes over time.

a) Compare their initial positions and speeds.

b) Describe how their motions differ.

Distance from Work



10. **Assessment Focus** The graduation committee is making arrangements for the prom. This graph shows the total cost of the prom based on the number of people who attend it.

a) What is the vertical intercept? What does it represent?

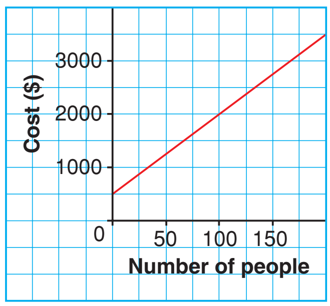
b) Calculate the rate of change in the cost with respect to the number of people. What does this rate of change represent?

c) The committee wants to sell tickets at \$20 per person. On a copy of the graph, sketch a line that shows the total sales.

d) How many tickets would have to be sold to break even?

e) Suppose the ticket price is \$25 per person. How would this change the sales graph? How would this change the break-even point? Justify your answers.

Cost of Prom



To *break even* means to neither make nor lose money.

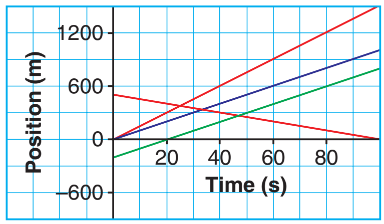
11. This graph shows the positions of four cars over time.

- Car A and Car B start from the same position, but travel at different speeds.
- Car C follows Car B at a constant distance behind.
- Car D travels in the opposite direction, passing each of the other cars in turn.

a) On a copy of the graph, label each line with the car it represents. Justify each choice.

b) Determine the speed of each car.

Motion of Four Cars



For help with questions 12 to 14, see Example 3.

12. The time gap between lightning and thunder occurs because sound travels more slowly than light. This table gives the speed of sound in air at various temperatures.

Temperature (°C)	0	10	20	30
Velocity (m/s)	332	338	344	350

a) Draw a scatter plot of the data and determine the equation of the line of best fit.

b) Predict the speed of sound at these air temperatures.

i) 15°C **ii)** 50°C **iii)** –3°C

13. Mass-for-age data were collected for two baby girls.

Age (months)	21	24	27	30	33	36
Alea's mass (lb.)	21.5	22.4	23.3	24.1	24.9	25.7
Yanxia's mass (lb.)	30.5	32.0	33.5	35.0	36.5	38.0

a) Do the babies' masses grow linearly? Explain.

b) Which baby is growing more quickly? Explain.

c) Find a linear regression equation that models the data for each baby.

d) How well does the line of best fit model the data? Explain.

e) Determine the rate of change in mass for each baby.

14. The table shows the distance travelled by a transport truck after each number of hours.

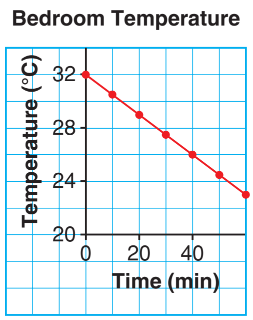
Time (h)	2	3	5	6
Distance travelled (m)	195	302	508	599

a) Find a linear regression equation that models the data.
b) Estimate the distance travelled in 4 hours. Why should this estimate be accurate?
c) Estimate the distance travelled in 24 hours. Why might this estimate be inaccurate?

15. **Literacy in Math** Make a Frayer Model for linear relations. Include *rate of change* in your model.

C

16. When a bedroom air conditioner is turned on High, the temperature in the bedroom drops as shown in this graph.



a) How fast is the temperature decreasing?
b) Sketch a graph for the situation when the initial temperature of the bedroom is greater. Explain your reasoning.
c) Sketch a graph for the situation where the air conditioner is on Low. Explain your reasoning.

17. Draw a graph to show how the price of gasoline changes with time.
- On January 1, the price is 95.0¢/L.
- The price increases 3.0¢/L each month for 4 months.
- The price decreases 1.5¢/L each month for 2 months.
- The price jumps to 115.0¢/L overnight.
- The price doesn't change for the rest of the year.

In Your Own Words

Why can we say that rate of change is the same as slope for linear graphs? Why is this not true for other graphs? Use examples to illustrate your answer.

5.4 Quadratic Models

Aram is an accident reconstructionist for the Ontario Provincial Police. He investigates car accidents in which serious injury or death occurred. Aram uses mathematical models to estimate the speed at which the vehicle was moving when it crashed.

Investigate

Investigating Relationships in Formulas

Materials
- TI-83 or TI-84 graphing calculator

Work with a partner.

The distance a vehicle travels in icy conditions, after the driver applies the brakes, can be modelled by the formula $d = 0.75sm^2$, where
- d metres is the distance travelled
- m tonnes is the vehicle's mass
- s kilometres per hour is the vehicle's speed

- Choose a reasonable speed s for the vehicle.
  Substitute into the formula $d = 0.75sm^2$ and simplify.
  Graph the resulting formula.
- Choose a reasonable mass m for the vehicle.
  Substitute into the formula $d = 0.75sm^2$ and simplify.
  Graph the resulting formula.

Reflect

What would be the effect of choosing a different value for s? For m?

Connect the Ideas

Quadratic models

In Lesson 5.3, we used linear models to represent a variety of real-world situations involving constant rates of change. In situations that do not involve constant rates of change, we must use a non-linear model. One possible non-linear model is a quadratic model.

A quadratic relation must have an x^2 term, so $a \neq 0$.

Quadratic models

Quadratic models have these characteristics.
- In a table of values, the second differences are equal.
- The graph is a curve called a *parabola*.
- The equation can be written in the form $y = ax^2 + bx + c$, $a \neq 0$.

Example 1 Identifying Quadratic Models

Which models represent quadratic relations?
Justify your answers.

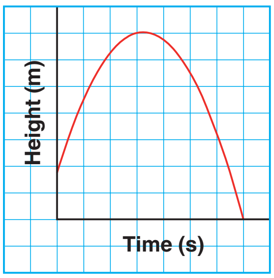
a)

h	0	1	2	3
p	250	238	202	142

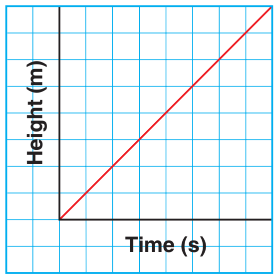
b)

r	0	1	2	3
Q	32	48	72	108

c)



d)



e) $y = x^2 + 7$

f) $y = 3x + 2$

Solution

a)

h	*p*	First differences	Second differences
0	250		
		$238 - 250 = -12$	
1	238		$-36 - (-12) = -24$
		$202 - 238 = -36$	
2	202		$-60 - (-36) = -24$
		$142 - 202 = -60$	
3	142		

Since the x-values increase by 1, the first differences are the rates of change.

The second differences are equal. The relationship is quadratic.

In a quadratic model, the first differences and rates of change are not constant.

In a linear model, the first differences and rates of change are constant.

b)

r	Q	First differences	Second differences
0	32		
		$48 - 32 = 16$	
1	48		$24 - 16 = 8$
		$72 - 48 = 24$	
2	72		$36 - 24 = 12$
		$108 - 72 = 36$	
3	108		

The second differences are not equal. The relationship is not quadratic.

c) The graph appears to be a parabola.
The relationship appears to be quadratic.

d) The graph is not a parabola.
The relationship is not quadratic.

e) Rewrite the equation as $y = 1x^2 + 0x + 7$. Since $a \neq 0$, the relationship is quadratic.

f) The equation has no x^2 term. So, the relationship is not quadratic.

Relationships in formulas

In *Investigate*, the formula $d = 0.75sm^2$ involved three variables: the dependent variable d, and two independent variables, s and m.

To investigate the relationship between d and s, we set m constant to get a formula involving only s and d. Similarly, to investigate the relationship between d and m, we set s constant to obtain a formula involving only d and m. We can use this strategy to investigate relationships in other formulas with more than two variables.

Example 2

Investigating Relationships in Formulas

Materials

- TI-83 or TI-84 graphing calculator

The formula $d = \frac{1}{2}at^2$ gives the distance travelled by a car as it accelerates from a stopped position; d metres is the distance travelled, a metres per second squared is the acceleration, and t seconds is the time elapsed.

a) Investigate the relationship between d and a when $t = 2$ s.

b) Investigate the relationship between d and t when $a = 2$ m/s^2.

c) What types of relationships were obtained in parts a and b? Why does it make sense that these relationships were obtained?

Solution

a) Substitute $t = 2$ in the formula.

$d = \frac{1}{2}at^2$

$d = \frac{1}{2}a(2)^2$

$d = 2a$

Enter the formula $d = 2a$ on a graphing calculator as $y = 2x$.

b) Substitute $a = 2$ in the formula.

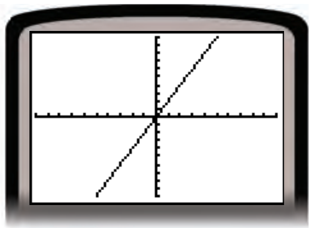
$d = \frac{1}{2}at^2$

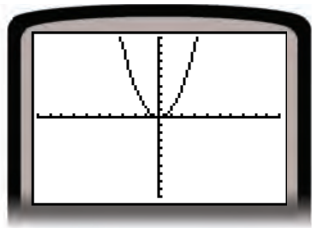
$d = \frac{1}{2}(2)t^2$

$d = t^2$

Enter the formula $d = t^2$ on a graphing calculator as $y = x^2$.

Press [ZOOM] 6 to view the equations using the standard window settings.





c) Setting t constant produces the linear relation $d = 2a$. This makes sense since a is raised to the exponent 1 in the formula $d = \frac{1}{2}at^2$. Setting a constant produces the quadratic relation $d = t^2$. This makes sense since t is raised to the exponent 2 in the formula $d = \frac{1}{2}at^2$.

Quadratic regression

In Lesson 5.3, we used linear regression to find the line of best fit. Similarly, we can use quadratic regression to find the parabola of best fit.

Example 3

Fitting a Quadratic Model to Data

Materials
- TI-83 or TI-84 graphing calculator

A fountain of sparks from a Canada Day rocket follows an arc in the air. This table shows the height of the sparks at various horizontal distances from the launching point.

Distance (m)	5	10	15	20	25	30
Height (m)	43	75	97	108	109	100

a) Determine the equation of the parabola of best fit.

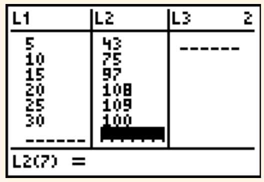
b) Determine the maximum of the regression curve. What does it represent?

Home Quit

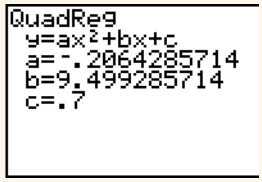
Solution

a) Use the **QuadReg** command.

Enter the data in lists **L1** and **L2**.

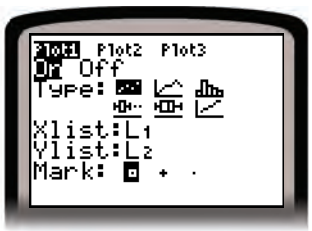


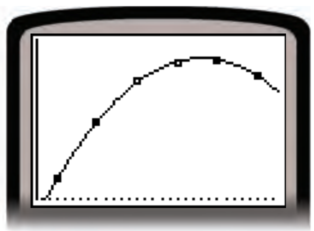
Press STAT ▶ 5 2nd 1 , 2nd 2 , VARS ▶ 1 1 ENTER to perform quadratic regression on the data and store the regression equation as **Y1**.



So, the regression equation is: $y \doteq -0.2064x^2 + 9.499x + 0.7$, where x represents the horizontal distance from the launching point and y represents the height of the sparks.

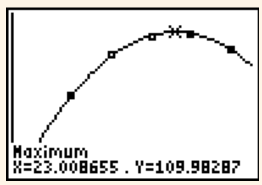
b) Graph the data and the regression curve.





Use the **maximum** feature.

Press: 2nd TRACE 4
Move the cursor to the left of the maximum and press ENTER, then to the right of the maximum and press ENTER, then close to the maximum and press ENTER.



The maximum of the regression curve is about 110.
It represents the maximum height of the sparks from the rocket: about 110 m

Practice

A

For help with questions 1 to 3, see Example 1.

1. Which table of values models a quadratic relation? How do you know?

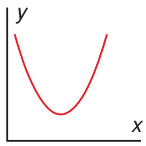
a)

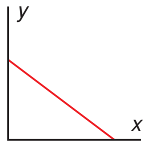
t	0	1	2	3	4
c	0.5	2	8	32	128

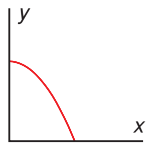
b)

t	0	1	2	3	4
c	0.5	5.5	20.5	45.5	80.5

2. Which graphs might model a quadratic relation? Why do you think so?

a)


b)


c)


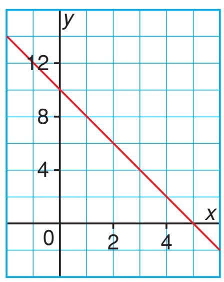
3. Which equations model a quadratic relation? How do you know?

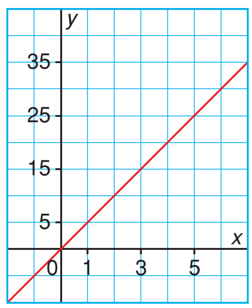
a) $y = -2x$ b) $y = x^2 + 1$ c) $y = 5 - 2x$

4. Match each equation with a graph.

a) $y = x + 3$ b) $y = x^2 - 4x + 3$ c) $y = 5x$

d) $y = -2x^2 + 2$ e) $y = -2x + 10$ f) $y = 4x^2 - 4x$

i)


ii)


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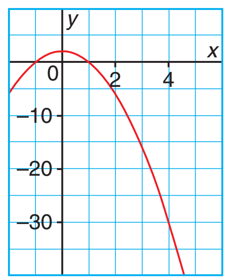

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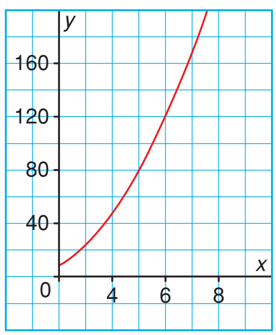

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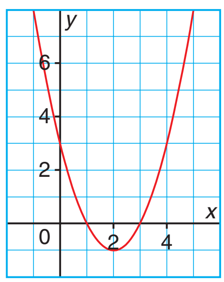

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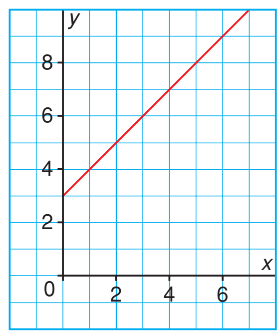


5. Tell whether each statement is true or false.

a) For a quadratic relation, the second differences are equal.

b) No quadratic relation can be represented by a parabola.

c) If the first differences are equal, the relation is quadratic.

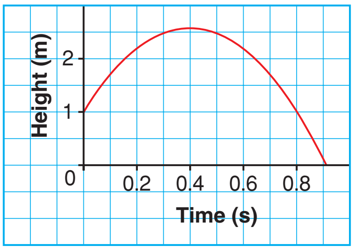
d) All relations with equal first differences are linear.

e) Some quadratic relations, but not all, can be represented by an equation of the form $y = mx + b$.

B

6. An orange is tossed straight up in the air.

Height of Orange



a) When is the height of the orange increasing? When is it decreasing?

b) When is the height of the orange changing rapidly? When is it changing slowly?

■ For help with questions 7 and 8, see Example 2.

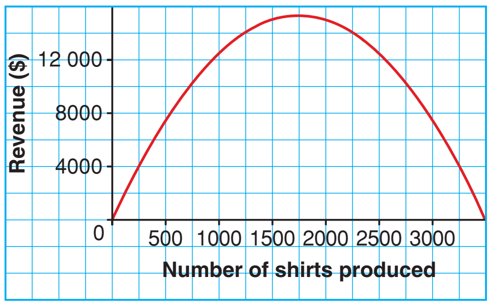
7. A T-shirt manufacturer wants to maximize her revenue.

a) Describe the trends in the graph.

b) As the number of shirts produced increases, why does the revenue increase and then decrease?

c) Explain how the manufacturer can use the graph to decide how to maximize her revenue.

Shirt Revenue



8. The formula for the volume of a cylinder with radius r and height h is:
$V = \pi r^2 h$

a) Which variable(s) in the formula $V = \pi r^2 h$ should you set constant to generate a linear relationship? Explain why you made that choice.

b) Which variable(s) in the formula $V = \pi r^2 h$ should you set constant to generate a quadratic relationship? Explain why you made that choice.

c) Verify your answers to parts a and b by graphing $V = \pi r^2 h$ when $r = 5$ cm and when $h = 5$ cm. Were you correct? Explain.

9. The volume V of a pyramid with height h and square base of side length s is given by the formula: $V = \frac{1}{3}s^2h$

a) Which variable should you set constant to generate a linear relationship?

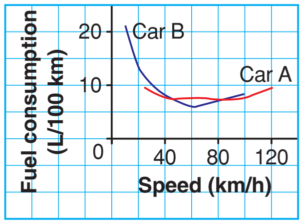
b) Which variable should you set constant to generate a quadratic relationship?

c) How can you check that your answers in parts a and b are correct? Explain.

■ For help with questions 10 to 12, see Example 3.

10. Assessment Focus This graph shows how the fuel consumptions of two vehicles change as their speed increases.

Fuel Efficiencies of Two Cars



a) Describe the trends in each graph.

b) When does the fuel efficiency for each vehicle increase or decrease rapidly? When does it increase or decrease at a constant rate? How do you know?

c) Which car is more fuel efficient at each speed?

i) 55 km/h **ii)** 110 km/h

d) Nyarai thinks the graph for car B appears to be quadratic. Do you agree or disagree? Justify your answer.

11. A juggler tosses balls from one hand to the other. These two equations represent the height of a ball, B metres above its release point, after t seconds.

Right-hand toss: $B = 8.4t - 9.8t^2$

Left-hand toss: $B = 3.4t - 9.8t^2$

a) Graph the equations on the same screen of a graphing calculator.

b) What is the maximum height of a ball tossed by each hand?

c) Assume that the ball is caught at the same height from which it was released. How long is a ball tossed by each hand in the air?

d) Which hand releases the ball at a faster speed? Justify your answer.

Average speed $= \dfrac{\text{Distance travelled}}{\text{Time taken}}$

12. The management of a hockey arena plans to increase ticket prices to obtain more revenue. A survey was conducted to estimate the revenue generated for different ticket prices. These data were obtained.

Ticket price ($)	Projected revenue ($)
10	16 000
15	19 500
20	20 300
25	14 750

a) Why might a quadratic model be a good fit for the data?
b) Perform a quadratic regression on the data.
c) Determine the maximum value of the regression equation. What does it represent?
d) Predict the revenue that would be obtained if the tickets cost $30.

13. a) Why might a quadratic model be a good fit for the data in this table?
b) Perform a quadratic regression on the data.
c) Determine the minimum value of the regression equation. What does it represent?
d) Predict the number of males registered in 2004. How does this compare with the actual value of 241 995?

Males Registered in Apprenticeship Programs in Canada	
Year	**Number of males**
1991	184 705
1992	172 740
1993	160 020
1994	153 275
1995	151 945
1996	152 840
1997	157 875
1998	161 595
1999	170 710
2000	181 610
2001	195 220
2002	209 650

14. Transit fares are growing as shown in the table.

Year	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Fare ($)	0.10	0.15	0.20	0.30	0.30	0.60	0.95	1.20	2.00	2.00	2.25

a) Determine the equation of the parabola of best fit.

b) Estimate the fare for each year.

i) 1987 **ii)** 2020

c) Which of your estimates in part b do you think is more accurate? Explain.

15. **Literacy in Math** Make a list of terms related to quadratic relations. Write a definition for each term. Use a sketch when helpful.

C

16. A grandfather clock that is running too fast or too slow can be fixed by adjusting the length of the pendulum. Use the formula $l = \frac{1}{4\pi^2}gT^2$, where:

- l metres represents the length of the pendulum
- T seconds represents the time to make one complete swing
- g m/s^2 is the acceleration due to gravity

On Earth, the value of g is 9.8 m/s^2.

a) Sketch a graph of length versus time.

b) Describe the relationship between length and time. Justify your answer in two different ways.

c) Imagine that the clock is taken to the Moon, where $g = 1.6$ m/s^2. How would the graph in part a change?

d) The clock is moved from planet to planet but always set so $T = 2$ s. Describe the relation between l and g.

17. Sketch a distance-time graph for the following race between the Tortoise and the Hare. Justify your graph.

- The Tortoise walks at a speed of 0.5 m/s for the entire 1-km race.
- The Hare runs fast and then more slowly, stopping after 50 s at the 100-m mark.
- The Hare then takes at nap.
- The Hare wakes up when the Tortoise is 10 m from the finish line.
- The Hare starts to run faster and faster but loses the race by 200 m.

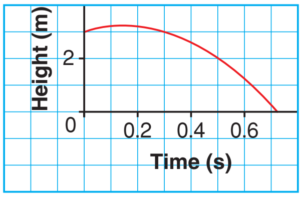
In Your Own Words

How are the steps involved in quadratic regression similar to steps for linear regression? How are they different? Explain.

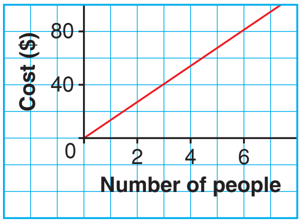
Mid-Chapter Review

5.1 **1.** Describe the relationship illustrated by each graph.

a) Diver Height

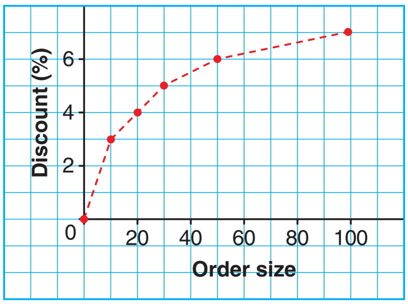


b) Museum Admission Cost



5.2 **2.** A computer store gives volume discounts.

Computer Volume Discount

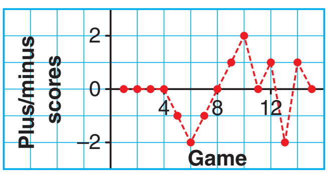


a) Describe the trends in the graph.

b) Predict the number of computers you would have to buy to get an 8% discount. Justify your prediction.

5.3 **3.** Determine when the rate of change in a hockey player's plus/minus score is zero, constant, or changing.

Hockey Player's Plus/Minus Scores



4. The median age, A, in each province n years after 1991 is:

Prince Edward Island: $A = 0.34n + 24$

Alberta: $A = 0.43n + 24$

a) Graph the equations on the same grid. How do the graphs compare?

b) How would the graph for Alberta change if the median age in 1991 was 30?

5.4 **5.** The volume V of a cone with height h and radius r is given by the formula $V = \frac{1}{3}\pi r^2 h$.

a) Which variable should you set constant to generate a linear relationship?

b) Which variable should you set constant to generate a quadratic relationship?

6. a) Why might a quadratic model be a good fit for the data in this table?

Year	Population of Kingston
2001	152 652
2002	154 439
2003	155 676
2004	156 123
2005	155 685
2006	154 971

b) Perform quadratic regression.

c) Do you think the trend modelled by the regression equation will continue? Explain.

Curves of Concentration

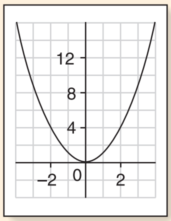
Materials

- *6 graph cards*
- *6 rate of change cards*
- *6 feature cards*

Play with a partner.

- Shuffle the cards. Spread them out face down so no cards overlap.
- One player turns over three cards.
  If the cards make a matching set, the player keeps them.
  If not, the player turns them over again.
- A matching set of cards contains each type of card.
  The feature and rate of change cards accurately describe the graph card. For example, this is a matching set:

Graph card



Rate of change card

The rate of change is negative, then positive.

Feature card

The graph has a minimum.

- Players take turns turning over the cards.
- The game ends when all cards have been picked up.
  The player with more cards wins.

Reflect

- Explain your strategy for deciding whether cards match.
- Compare strategies with your partner. What have you learned from your partner's strategy?

5.5 Exponential Models

You may have heard statements like these in the media:

- *The economy is projected to grow by 3.4% this year.*
- *College tuition fees are expected to rise by nearly 5% next year.*

Each of these situations involves a rate of change expressed as a percent.

Investigate: Comparing Constant Growth and Constant Percent Growth

Materials

- TI-83 or TI-84 graphing calculator

Work with a partner.

Suppose you are offered a choice of jobs.

- Job A pays \$10/h with a \$1/h raise every year.
- Job B pays \$10/h with a 10% raise every year.

- How would your wages grow under each job over 5 years? Organize your work in a table like this.
- Would you prefer to have Job A or Job B? Explain your reasoning.

Year	Job A (\$)	Job B (\$)
0	10	10
1		
2		
3		
4		
5		

Reflect

- Why do you think constant growth is also called additive growth?
- Why do you think constant percent growth is also called multiplicative growth?
- Why do your wages grow faster under Job B than under Job A?

Connect the Ideas

Linear and exponential models

- Linear models represent quantities that change at a *constant rate*; that is, a fixed amount is *added* to the quantity at regular intervals.
- Exponential models represent quantities that change at a *constant percent rate*; that is, the quantity is *multiplied* by a fixed amount at regular intervals.

Example 1

Recognizing Linear and Exponential Models

Last year, a school had a population of 1000 students. This year, 1100 students attend the school, an increase of 100 students or 10%.

a) Determine the population after 3 more years under each scenario.

- **Scenario A:** The population increases by 100 students each year.
- **Scenario B:** The population increases by 10% each year.

b) What type of growth does each scenario illustrate?

Solution

a) Use a table to record the growth in the student population.

A 10% increase means that you have 100% + 10% = 110% of the original amount. To find 110% of a number, multiply by 1.10.

	Student Population	
Year	**Scenario A: increase of 100**	**Scenario B: increase by 10%**
0	1100	1100
1	$1100 + 100 = 1200$	$1100 \times 1.10 = 1210$
2	$1200 + 100 = 1300$	$1210 \times 1.10 = 1331$
3	$1300 + 100 = 1400$	$1331 \times 1.10 \doteq 1464$

After 3 years, the student population is 1400 students under Scenario A and 1464 students under Scenario B.

b) Under Scenario A, we repeatedly added 100. This is linear growth. Under Scenario B, we repeatedly multiplied by 1.10. This is exponential growth.

Exponential models

Exponential models

Exponential models have these characteristics.

- In a table of values, the growth/decay factors are equal.
- The graph resembles an exponential curve.
- The equation can be written in the form $y = ab^x$, where a is the initial value and b is the growth/decay factor.

Example 2 Identifying Exponential Models

Which models represent exponential relations?
Justify your answers.

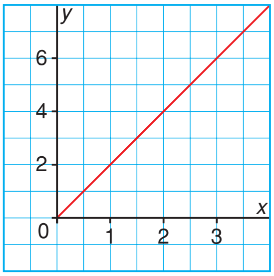
a)

t	0	1	2	3
A	35	25	15	5

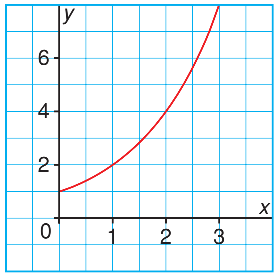
b)

d	0	1	2	3
P	51.2	64	80	100

c)



d)



e) $y = 10(2)^x$

f) $y = 10x^2$

Solution

a)

t	A	Decay factor
0	35	
		$\frac{25}{35} \doteq 0.71$
1	25	
		$\frac{15}{25} = 0.60$
2	15	
		$\frac{5}{15} \doteq 0.33$
3	5	

The decay factor is not constant. The relationship is not exponential.

b)

d	P	Growth factor
0	51.2	
		$\frac{64.0}{51.2} = 1.25$
1	64.0	
		$\frac{80.0}{64.0} = 1.25$
2	80.0	
		$\frac{100.0}{80.0} = 1.25$
3	100.0	

The growth factor is constant. The relationship is exponential.

c) The graph is a line. The relationship is not exponential.

d) The graph appears to be an exponential curve. The relationship appears to be exponential.

e) The equation is of the form $y = ab^x$. The relationship is exponential.

f) The equation is not of the form $y = ab^x$. The relationship is not exponential.

Example 3

Materials

- TI-83 or TI-84 graphing calculator

Comparing Pairs of Exponential Relations

Two colonies of bacteria each start with 100 bacteria.

- The population of Colony A doubles every hour.
- The population of Colony B triples every hour.

These equations represent the population, P bacteria, of the two colonies after t hours.

Colony A: $P = 100(2)^t$

Colony B: $P = 100(3)^t$

a) Graph the equations on the same screen. How do the graphs compare? Explain.

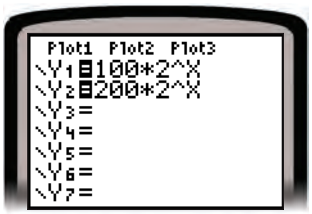
b) How would the graph for Colony A change if there were 200 bacteria initially?

Solution

a) Use a graphing calculator to generate the graphs of $y = 100(2)^x$ and $y = 100(3)^x$ on the same screen.

Both graphs have the same vertical intercept because each colony has the same initial number of bacteria, 100. The growth factor for Colony B is greater, so the curve for Colony B increases much faster than the curve for Colony A.

b) Generate the graphs of $y = 100(2)^x$ and $y = 200(2)^x$ on the same screen.



The vertical intercept would be 200 instead of 100.
Because the initial value would be greater, the graph would increase slightly faster than the original graph.

Exponential regression

We can use exponential regression to fit an exponential model to data.

Example 4 — Fitting an Exponential Model to Data

Materials
- TI-83 or TI-84 graphing calculator

a) Determine the exponential relation $y = ab^x$ that best fits the data in this table, where x is the number of years since 1921 and y is the population of British Columbia in millions.

Year	1921	1931	1941	1951	1961	1971	1981	1991	2001
B.C. Population (millions)	0.52	0.69	0.82	1.17	1.63	2.18	2.82	3.37	4.08

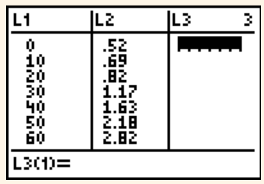
b) What do the values of a and b represent in this situation? Explain.

c) Estimate the population of British Columbia in 1985.

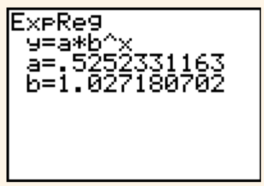
Solution

a) Use the **ExpReg** command.

Since x represents years since 1921, enter these values in list **L1:** 0, 10, 20, 30, 40, 50, 60, 70, 80. Enter the population data in list **L2**.



Press $\boxed{\text{STAT}}$ $\boxed{\blacktriangleright}$ 0 $\boxed{\text{2nd}}$ 1 $\boxed{,}$ $\boxed{\text{2nd}}$ 2 $\boxed{,}$ $\boxed{\text{VARS}}$ $\boxed{\blacktriangleright}$ 1 1 $\boxed{\text{ENTER}}$ to perform exponential regression on the data and store the regression equation as **Y1**.



So, the regression equation is: $y \doteq 0.5252(1.02718)^x$.

b) The initial value a represents the approximate population, about 530 000 people, of British Columbia in 1921. The growth factor b represents a growth rate in the population of about 2.7% per year.

c) The number of years between 1985 and 1921 is: $1985 - 1921 = 64$
Substitute $x = 64$ into the regression equation.

$y = 0.5252(1.02718)^x$

$\quad = 0.5252(1.02718)^{64}$, or about 2.9

The population was about 2.9 million people in 1985.

You could also use the **TRACE** feature or the command **Y1(64)** on your graphing calculator.

Practice

A

- For help with question 1, see Example 1.
- For help with questions 2 to 5, see Example 2.

1. A population grows by each percent per year.
By what factor is each year's population multiplied?

a) 3% **b)** 5% **c)** 12%

2. Which situations represent linear growth? Exponential growth?
Justify your answers.

a) A hairdresser increases the price of a haircut by $0.50 every year.

b) Gua's investment doubles every 20 years.

c) The players in each round of a tennis tournament are the winners from each pair in the previous round.

d) Daryl makes $10/h working as a line chef.

3. Which tables of values model an exponential relation?
How do you know?

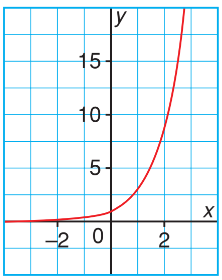
a)

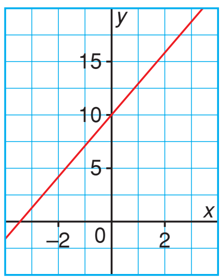
t	0	1	2	3	4	5
A	400	420	441	463	486.2	510.5

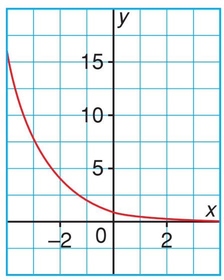
b)

d	0	1	2	3	4	5
P	100	82	67	55	45	37

4. Which graphs model an exponential relation? How do you know?

a)


b)


c)


5. Which equations model an exponential relation? How do you know?

a) $y = 2 + 4x$ **b)** $y = 2 + 4x^2$ **c)** $y = 2(4)^x$

B

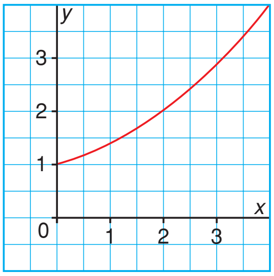
For help with questions 6 and 9, see Example 3.

6. For the given curve, sketch a different curve with:

a) A different starting value

b) A greater rate of change

c) A lesser rate of change



7. A city council wants to discourage illegal parking. It has two plans.

- **Plan A:** A \$10 fine for the first offence. The fine increases by \$20 for each subsequent offence.
- **Plan B:** A \$10 fine for the first offence. The fine doubles for each subsequent offence.

a) Determine the cost of a third fine under each plan.

b) What type of growth does each plan illustrate?

8. The half-life of penicillin in a patient with kidney disease is often longer than in a person without kidney disease. These equations represent the approximate concentration of penicillin, C micrograms per millilitre, in the bloodstream of each patient after t hours.

Patient with kidney disease: $C = 250(0.63)^t$

Patient without kidney disease: $C = 250(0.37)^t$

a) Graph the equations on the same screen. How do the graphs compare? Explain.

b) How would the graph for the patient with kidney disease change if the initial concentration of penicillin was 1000 μg/mL?

9. A formula to model the radioactive decay of Chromium-51 is $A = A_0(0.975)^t$, where A_0 is initial amount of Chromium-51 and A is the amount of Chromium-51 remaining after time t.

a) Which variable should you set constant to generate a linear relationship?

b) Which variable should you set constant to generate an exponential relationship?

c) How can you check that your answers to parts a and b are correct? Explain.

10. A general formula for population growth is $N = N_0(1 + r)^t$, where N_0 is the initial population, r is the growth rate as a decimal, and t is the time that has passed. Describe the shape of these graphs.

a) The graph of N against N_0 for fixed r and t

b) The graph of N against t for fixed r and N_0

Justify your answers.

For help with questions 11 and 12, see Example 4.

11. This table shows the growth in cell phone subscribers for a particular company.

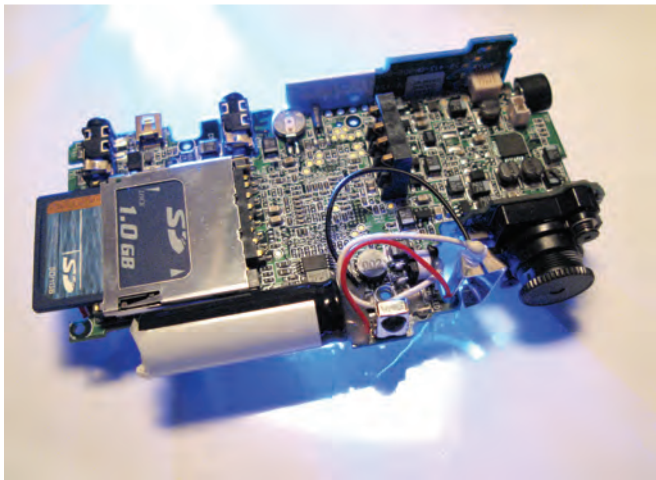
Year	2000	2001	2002	2003	2004
Number of subscribers (thousands)	15.9	33.8	43.9	55.3	86.1

a) Determine the exponential relation $y = ab^x$ that best fits the data, where x is the number of years since 2000 and y is the number of cell phone subscribers in thousands.

b) What do the values of a and b represent in this situation? Explain.

12. A camera stores charge in a capacitor and then uses it to create light when a flash is needed. This table shows the charge left in the capacitor after the flash begins.

Time (s)	Charge (μC)
0.00	100
0.02	82
0.04	67
0.06	55
0.08	45
0.10	37



a) Create a scatter plot of the data and describe any trends.

b) Determine the exponential relation $y = ab^x$ that best fits the data, where y is the charge left in the capacitor, in microcoulombs, after x seconds. Graph the regression curve on the scatter plot.

c) Estimate the charge left in the capacitor after each length of time.

i) 0.05 s **ii)** 0.20 s

13. **Assessment Focus** The volume of dough for cinnamon buns is measured at 10-min intervals.

Time (min)	0	10	20	30	40	50	60
Volume (L)	1.5	1.7	1.9	2.2	2.5	2.8	3.2

a) Create a scatter plot of the data and describe any trends.

b) Determine the exponential relation $y = ab^x$ that best fits the data, where y is the volume of dough, in litres, after x minutes.
Graph the regression curve on the scatter plot.

c) Estimate the volume of dough after each length of time.
i) 45 min ii) 90 min

d) Which estimate is likely to be more accurate? Explain.

e) How long do you think the trend in the data will continue? Give reasons for why it might change.

14. **Literacy in Math** Explain how you can determine whether a relationship is exponential. Include an example of a relationship that is exponential and an example of a relationship that is not exponential.

C **15.** The formula for the number of contestants C in a competition that began with B contestants and eliminates n contestants in each round r is $C = B\left(\frac{1}{n}\right)^r$. Is it possible to generate each type of relationship using this formula? If so, explain which variables you would set constant. If not, explain why not.

i) Linear
ii) Quadratic
iii) Exponential

In Your Own Words

Use an example to explain why an exponential relation has a constant growth or decay factor but a changing rate of change.

5.6 Selecting a Regression Model for Data

Diane is a quality control technician at a tire manufacturing company. She helps collect and analyse data to test how well the tires perform under different situations. Part of Diane's analysis involves finding a suitable model for the data.

Investigate

Comparing Regression Models

Materials

- TI-83 or TI-84 graphing calculator

Work in a group of 3.

This table shows how the air pressure in a car tire changes in the first 40 s after the tire is punctured.

Time (s)	Tire pressure (kPa)
0	207
5	186
10	145
15	110
20	90
25	62
30	48
35	41
40	28

- Have each person fit a different regression model to the data: linear, exponential, or quadratic.
- Compare the regression models. Use the model that you think best fits the data to predict the tire pressure:

  i) After 12 s **ii)** After 45 s

Reflect

How can you decide whether a linear, quadratic, or exponential relation provides the best fit for a set of data?

Connect the Ideas

Selecting a regression model

In Lessons 5.3 to 5.5, we were told whether to use linear, quadratic, or exponential regression to model data. In practical applications, we may not know what model to use. Examining first and second differences and growth or decay factors can help us make the best choice.

Example 1 Identifying Relationships in Data

Electrical appliances such as a VCR or a digital clock contain a capacitor for power during brief electrical outages. The table shows how the voltage in a capacitor decreases over time after a power outage.

Time (s)	0	2	4	6	8	10
Voltage (V)	9.0	7.0	5.2	3.9	3.0	2.3

What type of relationship seems to exist between voltage and time? Justify your answer.

Solution

Calculate the first differences, second differences, and decay factors.

Time (s)	Voltage (V)	First differences	Second differences	Decay factors
0	9.0			
		$7.0 - 9.0 = -2.0$		$\frac{7.0}{9.0} \doteq 0.8$
2	7.0		$-1.8 - (-2.0) = 0.2$	
		$5.2 - 7.0 = -1.8$		$\frac{5.2}{7.0} \doteq 0.7$
4	5.2		$-1.3 - (-1.8) = 0.5$	
		$3.9 - 5.2 = -1.3$		$\frac{3.9}{5.2} \doteq 0.8$
6	3.9		$-0.9 - (-1.3) = 0.4$	
		$3.0 - 3.9 = -0.9$		$\frac{3.0}{3.9} \doteq 0.8$
8	3.0		$-0.7 - (-0.9) = 0.2$	
		$2.3 - 3.0 = -0.7$		$\frac{2.3}{3.0} \doteq 0.8$
10	2.3			

The first and second differences are not equal, but the decay factors are approximately equal. There appears to be an exponential relationship between voltage and time.

Example 2

Selecting a Model to Represent Data

Materials

- TI-83 or TI-84 graphing calculator

In a science experiment, students punched a hole near the bottom of a 2-L pop bottle. They filled the bottle with water and measured how the water level changed over time. The results are shown in the table below.

Time (s)	0	25	50	75	100
Water level (cm)	30.0	22.3	16.1	11.2	7.8

a) Perform linear, quadratic, and exponential regressions on the data.

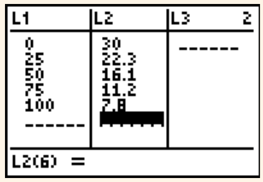
b) Which model best represents the data? Justify your answer.

c) Determine the height of the hole in the bottle. Justify your answer.

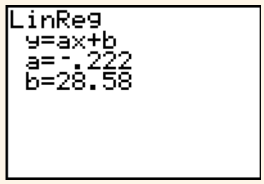
Solution

a) Perform each regression. Store the regression equations as **Y1, Y2,** and **Y3**.

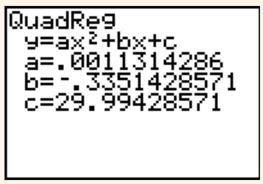
Enter data in lists **L1** and **L2**.



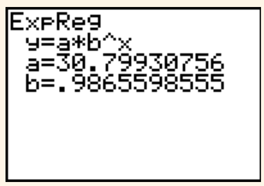
Press [STAT] [▶] 4 [2nd] 1 [,] [2nd] 2 [,] [VARS] [▶] 1 1 [ENTER] to perform linear regression on the data and store the regression equation as **Y1**.



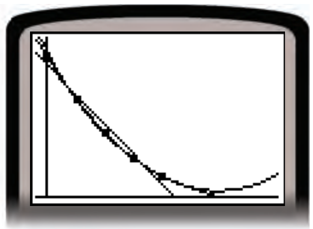
Press [STAT] [▶] 5 [2nd] 1 [,] [2nd] 2 [,] [VARS] [▶] 1 2 [ENTER] to perform quadratic regression on the data and store the regression equation as **Y2**.



Press [STAT] [▶] 0 [2nd] 1 [,] [2nd] 2 [,] [VARS] [▶] 1 3 [ENTER] to perform exponential regression on the data and store the regression equation as **Y3**.



b) Set up a scatter plot of the data. Press [ZOOM] 9 to view the scatter plot and the graphs of the regression equations.
Graph the data. Press [WINDOW] and set **Xmax** = 200 to show the minimum of the quadratic regression curve.

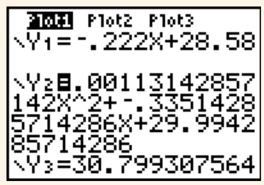


The linear model is clearly the worst fit for the data. Both the quadratic and exponential graphs look like a close fit for the data. Water below the hole cannot drain out of the bottle. So, the model that best represents the data will stop decreasing at some point. The graphs of the linear and exponential equations constantly decrease, while the graph of the quadratic equation has a minimum. So, a quadratic model best represents the data.

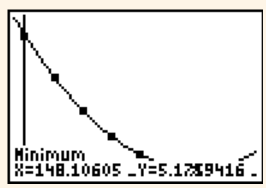
The quadratic relation is an accurate model of the data up to $t = 148$ s. After that, the water level will stay at 5.2 cm but the quadratic relation starts to increase. The quadratic model is not valid for later times.

c) Use the quadratic model. Graph the data and the regression parabola. Water will stop flowing out of the bottle when the water level is below the hole. So, the minimum of the regression parabola represents the height of the hole.

Press: [Y=]
Choose to display only the graph of the quadratic regression equation.
In row \Y1, highlight = and press [ENTER] to hide the linear graph.
Repeat for row \Y3 to hide the exponential graph.



Use the **minimum** feature.
Press: [2nd] [TRACE] 3
Move the cursor to the left of the minimum and press [ENTER], then to the right of the minimum and press [ENTER], then close to the minimum and press [ENTER].



The minimum of the regression parabola is about 5.2. The hole is about 5.2 cm above the bottom of the bottle.

Practice

A

For help with questions 1 to 3, see Example 1.

1. Use first differences to verify that a linear relation best represents each set of data.

a)

r	0	5	10	15	20
C	0	31	63	94	126

b)

x	0	1	2	3
y	10	12.5	15	17.5

2. Use second differences to verify that a quadratic relation best represents each set of data.

a)

v	0	1	2	3	4	5
E	0	0.5	2	4.5	8	12.5

b)

x	1	3	5	7
y	23	55	103	167

3. Use the growth/decay factors to verify that an exponential relation best represents each set of data.

a)

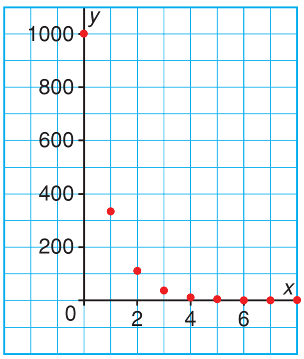
x	0	1	2	3
y	0.3	1.5	7.5	37.5

b)

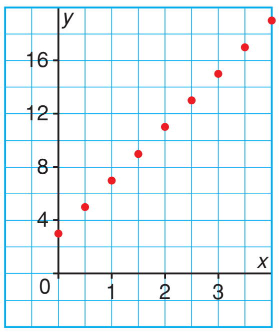
x	10	15	20	25
y	54	36	24	26

4. What type of regression should Jamil use for each set of data?

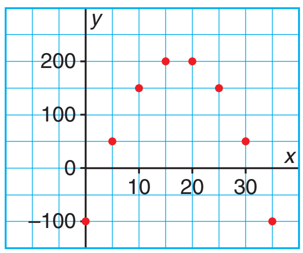
a)



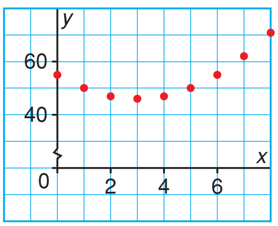
b)



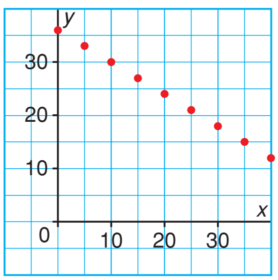
c)



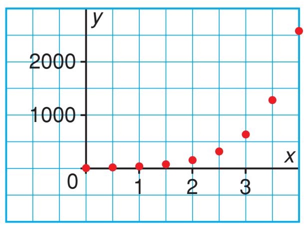
d)



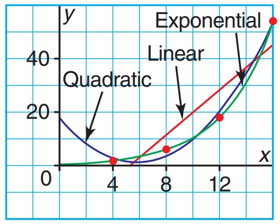
e)



f)



5. Toria fits a linear, quadratic, and exponential relation to a given set of data. Which relation do you think best models the data? Justify your answer.



6. Tell whether each set of data models a linear, quadratic, or exponential relation.

a)

x	0	1	2	3	4
y	0.5	2	8	32	128

b)

t	0	1	2	3	4
C	0.5	5.5	20.5	45.5	80.5

c)

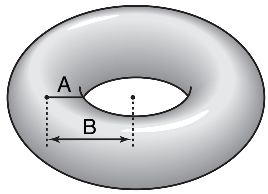
p	0	5	10	15	20
q	9.6	11.3	13.0	14.7	16.4

d)

v	3	6	9	12	15
w	11 250	2250	450	90	18

B

7. A company creates snow tubes with fixed radius B. Depending on how much air is pumped into the snow tube, radius A and volume change as shown in the table.



Radius A (cm)	15	16	17	18
Volume (cm^3)	73 282	83 378	94 126	105 526

a) What type of relationship seems to exist between radius A and volume? Justify your answer.

b) A snow tube is an example of a torus. The formula for the volume of this torus is $V = 2\pi^2 A^2 B$. Does this formula agree with the type of relationship you found in part a? Explain.

8. Rahul inherits money on his 25th birthday. He deposits it in an account that earns a constant rate of interest compounded annually. This table shows how much money he has after each year.

a) What type of relationship seems to exist between the amount and the number of years? Justify your answer.

Number of years	**Amount**
0	$6000.00
1	$6318.00
2	$6652.85
3	$7005.45
4	$7376.74

b) The formula for compound interest is $A = P(1 + i)^n$, where A is the amount, P is the principal invested, i is the interest rate, and n is the number of years. Does this formula agree with the type of relationship you found in part a? Explain.

9. A toy manufacturer makes two sets of plastic cylindrical cups. The cups in set A have the same height, but varying radii. The cups in set B have the same radius, but varying heights. These tables show the volumes of each set of cups.

Set A	
Radius (cm)	**Volume (cm³)**
1	23
2	92
3	207
4	368
5	575

Set B	
Height (cm)	**Volume (cm³)**
5	161
6	193
7	225
8	257
9	290

a) What type of relationship seems to exist between height and volume? Justify your answer.
b) What type of relationship seems to exist between radius and volume? Justify your answer.
c) The formula for the volume of a cylinder is $V = \pi r^2 h$, where V is the volume, r is the radius, and h is the height. How can you use this formula to verify your answers to parts a and b? Explain.

10. A diabetic on an insulin pump gives herself a bolus of insulin when she eats a meal or a snack. This table shows the units of insulin, in hundredths of a cubic centimetre, in the bolus depending on the number of carbohydrates she eats.

Carbohydrates (g)	50	60	70	80	90
Units of insulin	3.7	4.0	5.3	5.7	6.0

a) Perform linear, quadratic, and exponential regressions on the data.
b) Which model best represents the data? Justify your answer.

11. Verify your answers to question 6 by performing linear, quadratic, and exponential regressions on the data.

12. These data show the mass of algae in a pond during summer changes over time.

Time (s)	5	10	15	20	25	30
Mass (kg)	4.4	5.2	6.6	8.4	10.8	13.6

a) Perform linear, quadratic, and exponential regressions on the data.

b) Which model best represents the data? Justify your answer.

13. **Assessment Focus** These data show the speed of a rocket after it is launched.

Time (s)	5	10	15	20	25	30
Speed (m/s)	14	23	38	61	98	157

a) Create a scatter plot of the data.

b) Perform linear, quadratic, and exponential regressions on the data.

c) Which model best represents the data? Justify your answer.

d) How could you use the scatter plot to tell that a linear model would not be a good fit? Explain.

14. A 2-L bottle of pop is placed in a cooler filled with ice. The table shows how the temperature of the pop changes with time.

Time (h)	0	1	2	3	4
Temperature (°C)	26	11	4.8	2.1	1.0

a) Perform linear, quadratic, and exponential regressions on the data.

b) Which model best represents the data? Justify your answer.

c) Use the model you chose in part b to estimate the temperature of the pop after each length of time.

i) 6 h   ii) 24 h

d) Which estimate from part c do you think is more accurate?

e) Sketch a possible graph for temperature versus time for values of time from 0 h to 48 h. Explain your assumptions and the main features of the graph.

15. **Literacy in Math** Use a matrix or graphic organizer of your choice to compare linear, quadratic, and exponential relations. Include these categories: basic trends, first differences, second differences, growth/decay factors, and rate of change.

C

16. This table shows the number of hours of daylight in Dryden every 15 days.

Date	May 2	May 17	June 1	June 16	July 1	July 1	July 31
Hours of daylight	14.75	15.50	16.05	16.32	16.25	15.87	15.23

a) Create a scatter plot of the data.
b) Fit the data to an appropriate relation.
c) Explain how you chose which type of relation to use.
d) Estimate the number of hours of daylight on August 15.
e) Explain why it would be invalid to use your relation much past September 1.
f) Sketch a graph of the actual hours of daylight for a full year.

17. These data show the prices of different lengths of kitchen cabinet door handles.

Length (mm)	190	290	390	490	590	690	790	890	990
Price ($)	23.20	30.00	46.80	52.20	59.40	70.00	85.20	95.60	106.00

a) Create a scatter plot of the data.
Describe the trend in the data.
b) Perform linear, quadratic, and exponential regressions on the data.
c) Which model best represents the data?
Justify your answer.

In Your Own Words

Suppose you are given a set of data and asked to make a prediction based on the data. Describe how you would do this. What factors may affect the accuracy of the prediction?

5.7 Applying Trends in Data

Energy and the environment appear regularly in media headlines as society struggles to understand the issues and decide upon appropriate action. You can use what you have learned in this chapter to examine and interpret some of the data underlying these stories.

Inquire

Analysing Trends in Energy Usage and Glacier Melting

Materials

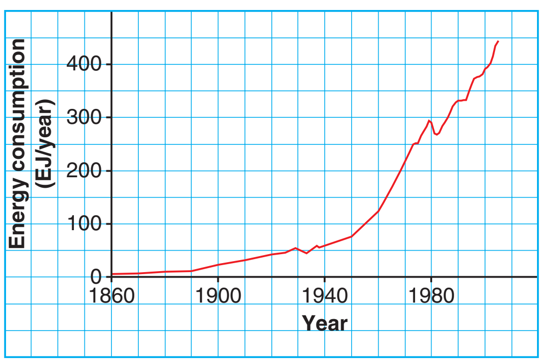
- TI-83 or TI-84 graphing calculator

Work in a small group.

Part A: World Energy Consumption

Scientists have recorded annual world energy consumption over the last 140 years.

Global Energy Consumption



1. Reading values from the graph

a) Create a table like this.

Year	Number of years since 1860	World energy consumption (EJ)
1860	0	
1870	10	
1990	130	
2000	140	

b) Read the world energy consumption values from the graph and enter them in the table.

2. Creating an exponential model

a) Describe the features of the graph that suggest that the growth in world energy consumption is approximately exponential.

b) Create a scatter plot of the data, and fit an exponential regression model to the data.
Record the regression equation.

c) Display the regression line and the scatter plot on the same screen. How well does the regression line model the data? Explain.

d) Use the model to predict the world energy consumption in 2015 and 2050.

3. Creating a linear model

a) Perform two linear regressions on the data: one for 1860 to 1950 and the other for 1950 to 2000.

b) Record the regression equations and draw the regression lines on the scatter plot from question 2.

c) Compare the fit of the two linear models to the fit of the exponential model.

d) Use the appropriate linear regression equation to predict the world energy consumption in 2015 and 2050. How do these predictions compare to those from the exponential model?

Part B: Glacier Melting

4. Choosing the best model to represent a set of data

This table shows the approximate area of the glaciers on Mount Kilimanjaro since 1880.

Year	1880	1912	1953	1976	1989	2003
Area (km^2)	20	12.1	6.7	4.2	3.3	2.5

a) Create a scatter plot of the data.

b) Perform linear, quadratic, and exponential regressions on the data.

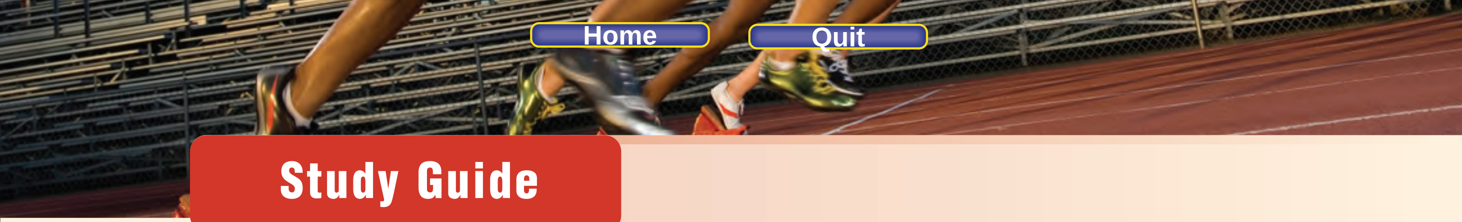
c) Which model do you think best estimates the trend in the data? Do you think the trend will continue? Explain.

d) Use each model to predict when the glaciers on Mount Kilimanjaro will disappear.

e) How similar are the predictions? Which prediction, if any, do you think is most accurate? Explain.

Reflect

- Why does it make sense to fit more than one regression model to the sets of data in this *Inquire*?
- In Part A, as you change models, how much do the predictions for both 2015 and 2050 change? Explain.
- Even if a regression model is an excellent fit to data, why might the actual future values be significantly different from those predicted by the model?



Study Guide

Rate of Change

- Average rate of change $= \dfrac{\text{Change in dependent variable}}{\text{Change in independent variable}}$

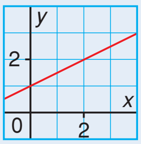
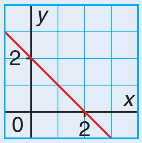
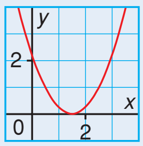
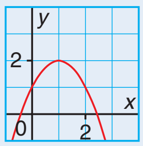
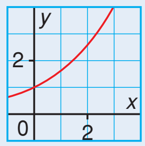
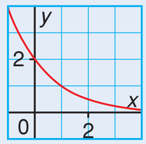
$$= \frac{\text{Change in temperature}}{\text{Change in time}}$$

$$= \frac{5°\text{C} - 20°\text{C}}{3\text{ h} - 1\text{ h}} = \frac{-15°\text{C}}{2\text{ h}} = -7.5°\text{C/h}$$

Temperature Change over Time

- If the graph is . . . then the rate of change
  - a horizontal line – is zero
  - a straight line – is constant
  - not a straight line – varies (changes)

Linear, Quadratic, and Exponential Models

Linear Graphs $y = mx + b$		**Quadratic Graphs** $y = ax^2 + bx + c$		**Exponential Graphs** $y = ab^x$	
$m > 0$	$m < 0$	$a > 0$	$a < 0$	$a > 0$ and $b > 1$	$a > 0$ and $0 < b < 1$
					
The first differences are constant.		The second differences are constant.		The decay/growth factors are constant.	

Regression

We can use the regression feature on a graphing calculator to fit a line or curve to data.

- Plot the data on a scatter plot.
- Choose a regression equation to model the data.

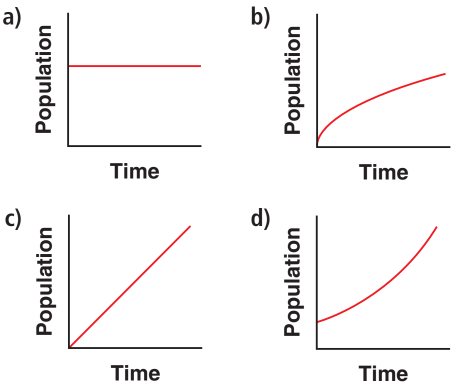
  Linear model: $y = ax + b$

  Quadratic model: $y = ax^2 + bx + c$

  Exponential model: $y = ab^x$
- Graph the regression equation on the scatter plot to see how well it fits the data.
- Use the regression equation to make predictions.

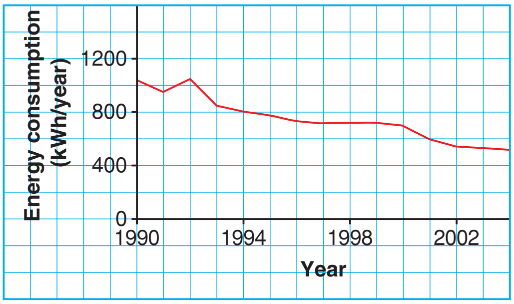
Chapter Review

5.1 **1.** Describe the relationship between population and time in each graph.



2. This graph shows the annual energy consumption of a new 22 cu. ft. refrigerator for each year.

Energy Consumption of a 22 cu. ft. Refrigerator



a) Describe the trends in the graph.

b) In which year were refrigerators the most inefficient? How do you know?

c) Predict the annual energy consumption of a new refrigerator in 2010. Explain your prediction.

5.2 **3.** Refer to the graphs in question 1.

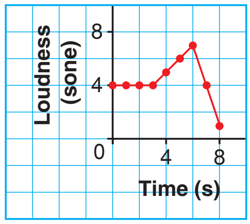
a) Which graph illustrates each rate of change?

i) Zero   ii) Constant   iii) Varying

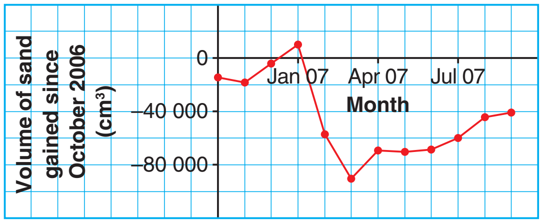
b) Identify the units of the rate of change for each graph.

4. Describe the rate of change indicated in each graph.

a) **Loudness of Television Program**

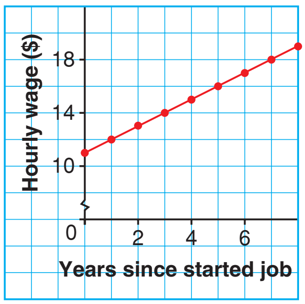


b) **Volume of Sand in a Beach**



5. This graph shows the growth in Zoltan's hourly wages.

Zoltan's Hourly Wage



a) Determine the rate of change in Zoltan's wages with respect to time.

b) What does the rate of change represent in this situation?

c) Does it matter which two points you use to calculate the rate of change? Explain.

5.3

6. Which of these tables of values models a linear relation? How do you know?

a)

s	0	10	20	30	40
t	62.4	58.6	54.8	51.0	47.2

b)

x	0	2	4	6	8
y	1	2	6	24	120

7. Kimiko and Atsuko have a catering business. The amount they charge, C dollars, depends on the event and the number of people, n, being served.

Lunch: $C = 100 + 15n$

Appetizers: $C = 100 + 10n$

a) In each formula, which number represents the rate of change? Explain.

b) What do the rates of change represent?

c) What other information does each formula give? Explain.

8. As China becomes more industrialized, its population consumes more plant oils.

Year	1991	1993	1997	2000	2004
Per person consumption (g/day)	23	25	29	33	35

a) Create a scatter plot of the data, and describe any trends.

b) Determine the equation of the line of best fit. Graph the line on the scatter plot.

c) Predict China's consumption of plant oils in 2010.

d) What factors could cause this consumption pattern to change? Explain.

5.4

9. Which of these equations models a quadratic relation? How do you know?

a) $y = 4x^2 + x - 1$

b) $y = 2x - 1$

c) $y = 3(2)^x$

d) $y = 3x^2$

10. The energy, E joules, of an object with mass m kilograms moving at a speed of v metres per second can be modelled by the formula $E = \frac{1}{2}mv^2$.

a) Use a graph to investigate the relationship between E and m when $v = 6$ m/s.

b) Describe the relationship between E and m. Justify your answer.

c) Use a graph to investigate the relationship between E and v when $m = 6$ kg.

d) Describe the relationship between E and v. Justify your answer.

11. A company designs custom prints for promotional beach balls. This table shows the surface areas of different balls.

Diameter (cm)	15	30	45	60	75
Surface area (cm^2)	707	2827	6362	11 310	17 671

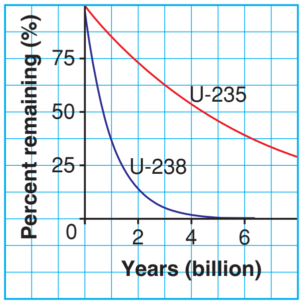
a) Create a scatter plot of the data, and describe any trends.

b) Determine the equation of the parabola of best fit. Graph the parabola on the scatter plot.

c) Predict the surface area of a ball with diameter 122 cm.

5.5 **12.** Use this graph.

Percent of Uranium Isotopes Remaining Since Their Creation



a) Compare the initial quantities of each radioactive isotope. What do you notice? Explain why this makes sense.

b) Which isotope has the greater decay factor? Explain.

13. Nyarai deposits \$2500 in an account that earns a constant percent interest compounded annually. This table shows the amount in the account after 4 years.

Year	Amount
0	\$2500.00
1	\$2650.00
2	\$2809.00
3	\$2977.54
4	\$3156.19

a) Create a scatter plot of the data, and describe any trends.

b) Determine the equation of the exponential regression curve. Graph the curve on the scatter plot.

c) Predict the amount in Nyarai's account after 10 years.

5.6 5.7 **14.** Identify which type of relation each equation represents.

a) $y = 5x + 2$

b) $y = 5(2)^x$

c) $y = 5x^2$

d) $y = x^2 + x + 6$

e) $y = 1.8x$

f) $y = -2x - 7$

15. The average house price in a popular neighbourhood has been increasing.

Year	2001	2002	2003	2004	2005
Price (thousands of dollars)	364	464	547	594	644

a) Draw a scatter plot of the data.

b) What type of relationship seems to exist between the average house price and the year? Justify your answer.

c) Use the model you chose in part a to generate a regression equation for the data, where y represents the average price of a house x years after 2000.

d) Estimate the average house price in 1995, 2000, and 2010. What factors may affect the reliability of these estimates?

16. Natural gas production in Norway has been increasing with the development of North Sea projects.

Year	2000	2001	2002	2003	2004
Production (million tonnes)	43	48	57	64	76

a) Draw a scatter plot of the data.

b) Perform linear, quadratic, and exponential regressions on the data.

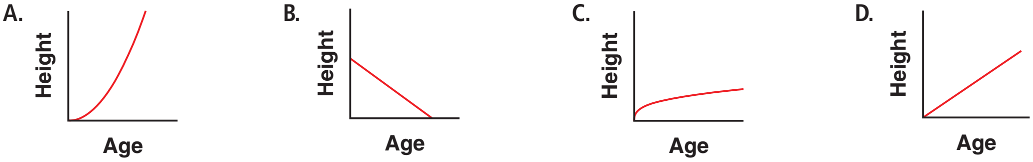
c) Which model best represents the data? Justify your answer.

d) Predict the natural gas production in Norway in 2010 and 2020. What factors may affect the accuracy of your predictions?

Practice Test

Multiple Choice: Choose the correct answer for questions 1 and 2. Justify each choice.

1. Which of these graphs best represents the relationship between a person's age and height?



2. Which variable in $d = \frac{1}{2}at^2$ should be set constant to obtain a linear relation?

A. d **B.** T **C.** a **D.** t

3. **Knowledge and Understanding** This table shows Raluca's height at different ages. What was the average rate of change in her height?

Age (months)	Height (cm)
3	59.8
9	70.1

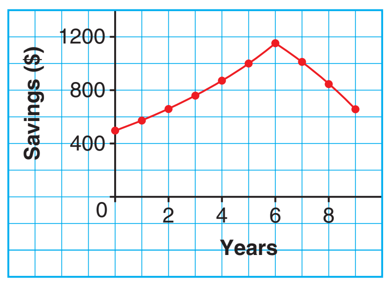
4. **Thinking** Jason deposited money in a savings account. He let the money earn interest for 6 years, and then made regular withdrawals. Explain how the graph would change under each scenario. Include a sketch in your explanation.

i) The account earned a lesser interest rate.

ii) Jason made greater regular withdrawals.

iii) Jason deposited a greater principal.

Jason's Savings



5. **Application** The United States imports aluminum cans from Canada for recycling.

a) Select a regression model to represent the trend in the data. Justify your selection.

b) Use your regression model from part a. Predict the mass of cans imported in 2007.

Year	1995	1997	1999	2001	2003	2005
Mass of cans imported (thousand tonnes)	27.2	34.4	41.5	43.3	47.1	55.4

6. **Communication** This graph compares the mean weight change over time for three popular diets.

a) Describe the trends in each graph.

b) Suppose you planned to start a diet. Which diet would you choose? Explain.

Average Weight Loss over Time on Different Diet Plans

Chapter Problem Modelling the Price of Diamonds

Diamonds are graded by the five C's: carats, clarity, cut, colour, and certification. A carat is a measure of the size of the diamond. So, for diamonds with similar clarity, cut, colour, and certification, the cost will depend on the size.

Size (carats)	Cost ($)
0.23	610
0.25	639
0.28	679
0.30	821
0.32	899
0.39	1056
0.42	1136
0.46	1104
0.74	2879
1.01	4066
1.04	5618

Decide what tools you can use to represent data.

1. Bella thinks a linear relation is best to predict the price of a diamond. Why might she think this?

2. Max thinks a quadratic relation is best to predict the price of a diamond. Why might he think this?

3. Carmen thinks an exponential relation is best to predict the price of a diamond. Why might she think this?

4. Which do you think is best for modelling the relation between the size in carats and the cost of a diamond: linear, quadratic, exponential relation? Justify your answer.